1. Deutsch-Jozsa problem
2. Ciruit transformations
3. Measurement techniques
4. Quantum teleportation

1. $f : \{0,1\}^n \longrightarrow \{0,1\}$

Promise:
f is either constant or balanced $|f^{-1}(0)| = |f^{-1}(1)|$

Q: which is it?

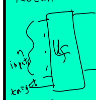
Classical queries to f
$f(0^n)$? ...

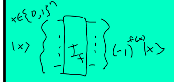
Worst-case: need $2^{n-1}+1$ queries to f to know for sure.

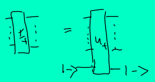
Quantum circuit that gives the answer with certainty with one query to f.

Recall

$U_f|\overbrace{x_1 \cdots x_n}^{x}, t\rangle = |x_1 \cdots x_n, t \oplus f(x)\rangle$







$|-\rangle = H|1\rangle = HX|0\rangle$

Run thru circuit on right:

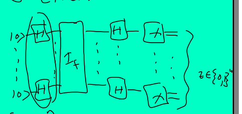
$|x\rangle \otimes |-\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle \otimes (|0\rangle - |1\rangle)\right)$

$\xmapsto{U_f} \frac{1}{\sqrt{2}}\left(|x\rangle \otimes (|f(x)\rangle - |\overline{f(x)}\rangle)\right)$

$= \begin{cases} |x\rangle \otimes |-\rangle & \text{if } f(x) = 0 \\ -|x\rangle \otimes |-\rangle & \text{if } f(x) = 1 \end{cases}$

$= (-1)^{f(x)} |x\rangle \otimes |-\rangle$

D-J circuit:



$x \in \{0,1\}^n \quad x = x_1 \cdots x_n$

$H^{\otimes n}|x\rangle = H|x_1\rangle \otimes H|x_2\rangle \otimes \cdots \otimes H|x_n\rangle$

$= \frac{1}{2^{n/2}}(|0\rangle + (-1)^{x_1}|1\rangle) \otimes \cdots$
$\otimes (|0\rangle + (-1)^{x_n}|1\rangle)$

$= \frac{1}{2^{n/2}} \sum_{y \in \{0,1\}^n} (-1)^{x_1y_1}(-1)^{x_2y_2} \cdots |y\rangle \quad y = y_1 \cdots y_n$

$= \frac{1}{2^{n/2}} \sum_y (-1)^{x_1y_1 + \cdots + x_ny_n} |y\rangle$

$= \frac{1}{2^{n/2}} \sum_y (-1)^{x \cdot y} |y\rangle$

Run the circuit:

$|0^n\rangle = |0\rangle^{\otimes n}$

$\xmapsto{H^{\otimes n}} \frac{1}{2^{n/2}} \sum_{y \in \{0,1\}^n} (-1)^{0^n \cdot y} |y\rangle$

$= \frac{1}{2^{n/2}} \sum_y |y\rangle$

$\xmapsto{I_f} \frac{1}{2^{n/2}} \sum_y I_f |y\rangle$

$= \frac{1}{2^{n/2}} \sum_y (-1)^{f(y)} |y\rangle$

$\xmapsto{H^{\otimes n}} \frac{1}{2^{n/2}} \sum_y (-1)^{f(y)} \left(H^{\otimes n}|y\rangle\right)$

$= \frac{1}{2^n} \sum_y (-1)^{f(y)} \left(\sum_{z \in \{0,1\}^n} (-1)^{y \cdot z} |z\rangle\right)$

$= \frac{1}{2^n} \sum_{y,z} (-1)^{f(y) + y \cdot z} |z\rangle$

$\xmapsto{\text{measure}}$?

Case 1: f is constant:

$\pm \frac{1}{2^n} \sum_{y,z} (-1)^{y \cdot z} |z\rangle$

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$$\pm \frac{1}{2^n} \sum_{y,z} (-1)^{y \cdot z} |z\rangle$$

$$= \pm \frac{1}{2^n} \sum_{z} \left(\sum_{y} (-1)^{y \cdot z} \right) |z\rangle$$

$$= \pm |0^n\rangle$$

measure: get all 0's with probability 1.

Case 2: f is balanced.

Final state is

$$\frac{1}{2^n} \sum_{z} \left(\sum_{y} (-1)^{f(y) + y \cdot z} \right) |z\rangle$$

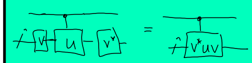
$$= \frac{1}{2^n} \left(\underbrace{\sum_{y} (-1)^{f(y)}} \right) |0^n\rangle + \cdots$$

$$= \sum_{z \neq 0^n} \cdots$$

Measure z, never see 0^n (instead some random string $\neq 0^n$)

If measurement gets 0^n,
 output "constant"
else
 output "balanced"

2. U, V n-qubit unitaries

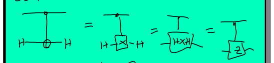


In particular:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ is Hermitian & unitary $H^2 = I$

Check: $HXH = Z$
$HZH = X$

So:



$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$C\text{-}Z_{1,2} |a,b\rangle = (-1)^{a \wedge b} |a,b\rangle$

$= C\text{-}Z_{2,1}$

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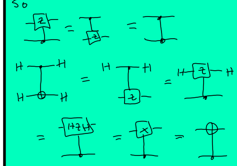
So



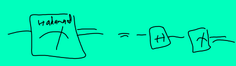
Btw: $HYH = -Y$

3. Measuring

$P_+ := HP_0H$
$P_- := HP_1H$

Hadamard basis measurement

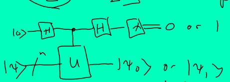
$|+\rangle \xmapsto{\text{measure}} 0$
$|-\rangle \longmapsto 1$



Hadamard test

U unitary & Hermitian, so $U^2 = I$.

$\left\{ \frac{I+U}{2}, \frac{I-U}{2} \right\}$ is a csop



0 means projector $\frac{I+U}{2}$
1 " " $\frac{I-U}{2}$

So $|\psi_0\rangle$ is in the image of $\frac{I+U}{2}$
$|\psi_1\rangle$ " - - - - - $\frac{I-U}{2}$

equivalently,
 $|\psi_0\rangle$ is an eigenvector of U with eigenvalue +1
 $|\psi_1\rangle$ is - - - - - U
 - - - - - -1