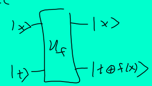
1. Deutsch's Problem (cont.)
2. Classical states, gates, circuits
3. Spectral Decomp^n (time permitting)

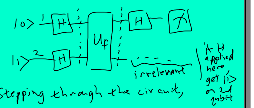
1. $f : \{0,1\} \to \{0,1\}$

f-gate

$t, x \in \{0,1\}$



Deutsch's "circuit"



Stepping through the circuit,

$$|01\rangle \xmapsto{H_1 H_2} H|0\rangle \otimes H|1\rangle$$
$$= \frac{1}{2}\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle - |1\rangle\right)$$
$$= \frac{1}{2}\Big(|0\rangle \otimes (|0\rangle - |1\rangle) + |1\rangle \otimes (|0\rangle - |1\rangle)\Big)$$
$$\xmapsto{U_f} \frac{1}{2}\Big(|0\rangle \otimes \left(|f(0)\rangle - |\overline{f(0)}\rangle\right) + |1\rangle \otimes \left(|f(1)\rangle - |\overline{f(1)}\rangle\right)\Big)$$
$$\xmapsto{H_1} \frac{1}{2\sqrt{2}}\Big(\left(|0\rangle + |1\rangle\right) \otimes \left(|f(0)\rangle - |\overline{f(0)}\rangle\right) + \left(|0\rangle - |1\rangle\right) \otimes \left(|f(1)\rangle - |\overline{f(1)}\rangle\right)\Big)$$
$$= \underbrace{\frac{1}{2\sqrt{2}}\Big(|0\rangle \otimes \left(|f(0)\rangle - |\overline{f(0)}\rangle + |f(1)\rangle - |\overline{f(1)}\rangle\right) + |1\rangle \otimes \left(|f(0)\rangle - |\overline{f(0)}\rangle - |f(1)\rangle + |\overline{f(1)}\rangle\right)\Big)}_{|\psi\rangle}$$

Case 1: f is constant: $f(0) = f(1)$

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\Big(|0\rangle \otimes \left(|f(0)\rangle - |\overline{f(0)}\rangle + |f(0)\rangle - |\overline{f(0)}\rangle\right) + 0\Big)$$
$$= \frac{1}{2\sqrt{2}}\Big(|0\rangle \otimes \left(2|f(0)\rangle - 2|\overline{f(0)}\rangle\right)\Big)$$
$$= |0\rangle \otimes \left(\frac{|f(0)\rangle - |\overline{f(0)}\rangle}{\sqrt{2}}\right)$$

measuring 1st qubit, get 0 with certainty.

Case 2: f is balanced, $f(1) = \overline{f(0)}$:

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\Big(\underbrace{|0\rangle \otimes (0)}_{0} + |1\rangle \otimes \left(2|f(0)\rangle - 2|\overline{f(0)}\rangle\right)\Big)$$

measure qubit 1, get 1 with certainty.

2. An n-qubit quantum state is classical means it is a computational basis state: $|x\rangle$ for some $x \in \{0,1\}^n$

An n-qubit unitary operator (gate) is classical means it maps classical states to classical states.

Quantum ops that are classical must be reversible

Matrix form M of a classical gate is a permutation matrix:

To be classical, M must map e_j to some e_k:

$$\begin{array}{r} \\ \\ \\ k \to \\ \\ \\ \\ \end{array}\underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{j\text{th col of } M} = \underbrace{\begin{bmatrix} & & & & \\ & & & & \\ & & & & \end{bmatrix}}_{M} e_j$$

$\therefore$ M has n many 1's exactly.

But if 2 cols have 1 in the same row, then some row is all 0's (pigeonhole principle)

$\therefore$ M is not invertible ($\det M = 0$),

$\therefore$ This can't happen

$\therefore$ M has exactly one 1 is every row and every column.

(def of a permutation matrix)

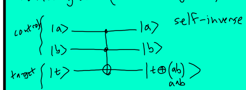
$\therefore$ $M^* = M^T$ — also a permutation matrix

How quantum circuits can do classical computations

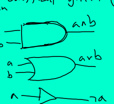
Primitive classical gates:

Pauli X (classical NOT)

Toffoli gate (C-C-X gate)

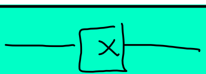


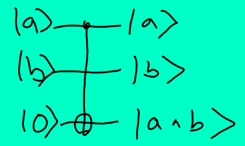
Classical classical gates (boolean gates):
non quantum

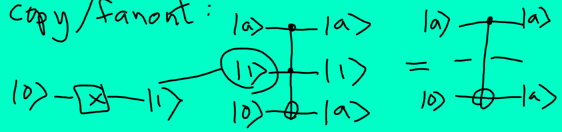


One more quantum primitive preparing a qubit in the $|0\rangle$ state.

Neet to simulate AND & NOT gates quantumly.

NOT: 

AND: 

copy/fanout: 

with copy, AND, NOT, can simulate any classical boolean circuit

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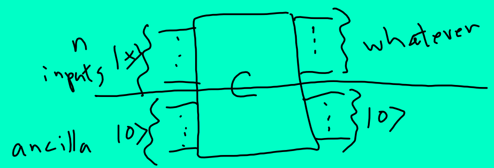
Simulating a boolean circuit reversibly usually requires extra wires that are not input wires: ancilla qubits.

Want modularity with our quantum circuits:

Def: A quantum circuit is clean if, assuming all ancilla qubits start in the $|0\rangle$ state, these qubits all return to the $|0\rangle$ state at the end of the circuit



C corresponds to a unitary gate on the input qubits only. (ancilla used only "locally" or temporarily.) Without cleanliness, info can bleed into the ancilla, so output is entangled with the ancilla — info loss on the output.

$f: \{0,1\}^n \longrightarrow \{0,1\}$

Want clean computation of f. (with just unitary gates):