Spectral Theorem (cont.) & consequences

Some quantum circuits

<u>Spectral Theorem</u> (unitary conjugation version). For every normal $A \in \mathbb{C}^{n\times n}$, there exists a diagonal matrix D and unitary U (both in $\mathbb{C}^{n\times n}$) such that

$$A = UDU^*.$$

Further, the entries of D are unique up to ordering. They comprise the spectrum of A.

<u>Proof</u>: Induction on n.

$n=1$ trivial: $A = [a]$ is diagonal.

$n>1$: $\mathrm{char}_A(\lambda)$ has degree n, so has a root λ, an eigenvalue of A. So there is a corresponding eigenvector $b_1 \in \mathbb{C}^n$:

$$Ab_1 = \lambda b_1.$$

Let W be any unitary matrix in $\mathbb{C}^{n\times n}$ that maps b_1 to e_1:

$$Wb_1 = e_1$$

Consider WAW^*.

$W^* e_1 = b_1$, so

$$WAW^* e_1 = WAb_1 = \lambda W b_1 = \lambda e_1$$

WAW^* is normal, and

$$WAW^* = \left[\begin{array}{c|c} \lambda & \\ \hline 0 & \\ \vdots & \\ 0 & \end{array}\right]$$

Since WAW^* is normal then by the lemma proved last time

$$WAW^* = \left[\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \underset{=V'D'V'^*}{B} \\ 0 & \end{array}\right]$$

and $B \in \mathbb{C}^{(n-1)\times(n-1)}$ is also normal.

By the inductive hypothesis there exists an $(n-1)\times(n-1)$ unitary V' and diagonal matrix D' such that

$$B = V'D'V'^*.$$

Let

$$V := \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & V' \\ 0 & \end{array}\right] \in \mathbb{C}^{n\times n}$$

Then V is unitary (check!) and

$$WAW^* = \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & V'D'V'^* \end{array}\right]$$

$$= V \left[\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & D' \\ 0 & \end{array}\right] V^* \quad \text{(check!)}$$

Letting $D := \left[\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline 0 & \\ \vdots & D' \\ 0 & \end{array}\right]$

D is diagonal and

$$VDV^* = WAW^*$$

$$\therefore\ W^*VDV^*W = W^*WAW^*W$$

$$(W^*V)D(W^*V)^* = A$$

Set $U := W^*V$.

U is unitary and

$$A = UDU^* \quad \blacksquare$$

For uniqueness:

$$\mathrm{char}_A = \mathrm{char}_{U^*AU} = \mathrm{char}_D$$

roots of char_A = eigenvalues of A

roots of char_D = " " D = diagonal elements of D

Apps of the spectral thm

Hermitian operators (including positive semidefinite operators, density matrices, projectors) are all normal.

Unitary operators are all normal.

projectors:

P projector

$P = P^* P$

Let U be unitary such that $P = UDU^*$ for diagonal D.

$$UDU^* = P = P^*P = (UDU^*)^*(UDU^*)$$
$$= UD^*\underline{U^*U}DU^* = UD^*DU^*$$

$D = \mathrm{diag}(d_1, \ldots, d_n)$ $(d_i \in \mathbb{C})$

$D^* = \mathrm{diag}(d_1^*, \ldots, d_n^*)$

$$D^*D = \mathrm{diag}(d_1^* d_1, \ldots, d_n^* d_n)$$
$$= \mathrm{diag}(|d_1|^2, \ldots, |d_n|^2)$$

But $D = D^*D$, so each $d_j = |d_j|^2$ for $1 \le j \le n$

$\therefore d_j \ge 0$ so $d_j = |d_j|$

so $d_j = d_j^2$

$d_j^2 - d_j = 0$

$d_j(d_j - 1) = 0$

so $d_j \in \{0,1\}$

$\therefore$ P is unitary conj to a diagonal matrix with 0's & 1's along the diagonal,

WLOG $P = U\left[\begin{array}{ccc|ccc} 1 & \ddots & 0 & & & \\ 0 & & 1 & & 0 & \\ \hline & & & 0 & \ddots & 0 \\ & 0 & & & & 0 \end{array}\right]U^*$

$$= U\left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]U^*$$

$\mathrm{spec}(P) \subseteq \{0,1\}$

Hermitian operators are those normal operators with spectrum $\subseteq \mathbb{R}$ (proof similar to the above

Unitary operators are those normal operators whose eigenvalues have unit norm (on the unit circle in $\mathbb{C}$

[proof is similar]

Theorem: Let $\{A_i\}_{i\in\mathcal{I}}$ be any collection of normal operators (indexed by an arbitrary set $\mathcal{I}$) such that

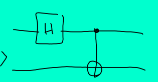
$A_i A_j = A_j A_i$ for all $i,j \in \mathcal{I}$.

There exists a unitary U such that $\forall j \in \mathcal{I}$, UA_jU^* is diagonal. Equiv there is a common eigenbasis for all the A_j

[Proof in the notes.]

Some quantum circuits

$|a\rangle$



converts comp. basis states to Bell stat

$|b\rangle$

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$=: |$

$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

If $b \in \{0,1\}$ ...

$$|ab\rangle \xmapsto{H_1} \left(\frac{|0\rangle + (-1)^a|1\rangle}{\sqrt{2}}\right) \otimes |b\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0b\rangle + (-1)^a |1b\rangle\right)$$

$$\xmapsto{C\text{-}X_{1,2}} \frac{1}{\sqrt{2}}\left(C\text{-}X_{1,2}|0b\rangle + (-1)^a C\text{-}X_{1,2}|1b\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0b\rangle + (-1)^a |1\bar{b}\rangle\right)$$

$$\left[\bar{b} := 1-b := \neg b\right]$$

Letting $B := (C\text{-}X_{1,2})H_1$

[operator given by the circuit]

$$B|ab\rangle = \frac{1}{\sqrt{2}}\left(|0b\rangle + (-1)^a|1\bar{b}\rangle\right).$$

$$B|00\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) =: |\Phi^+\rangle$$

$$B|01\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) =: |\Psi^+\rangle$$

$$B|10\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) =: |\Phi^-\rangle$$

$$B|11\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) =: |\Psi^-\rangle$$

Deutsch's Problem:

$f : \{0,1\} \longrightarrow \{0,1\}$

4 possibilities for f:

- f is constant 0
- " " " 1

} f is constant

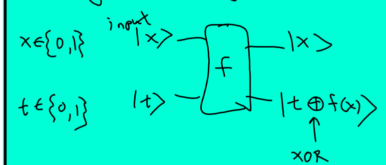
- " is the identity
- " is negation

} f is <u>balanced</u>

Classically, need 2 queries to f (in the worst case) to determine whether f is constant or balanced.

A quantum circuit only needs one query to determine this with certainty.

A quantum query to f means using an "f-gate"



This is a classical unitary gate

$=: U_f$

$$U_f U_f = I \quad \& \quad U_f = U_f^*$$