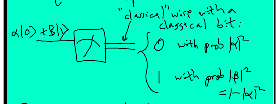
## Measurement gates

## Normal operators, diagonal operators, & the spectral theorem

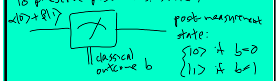
A measurement gate

is a computational basis projective measurement on one qubit. Depicted like




Post-measurement state is ignored/discarded

To preserve post-meas. state:




Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, pre-measurement state

$$\text{Prob}[0] = \langle\psi|P_0|\psi\rangle \quad P_0 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

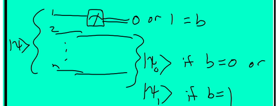
$$\text{Prob}[1] = \langle\psi|P_1|\psi\rangle$$

$$\langle\psi|P_0|\psi\rangle = \underbrace{\begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$= \begin{bmatrix}\alpha^* & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha^*\alpha\end{bmatrix} = |\alpha|^2$$

Sim for Prob[1].

n-qubit register




Prob[b] =? Generally,

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle \quad (\alpha_x \in \mathbb{C} \text{ satisfying } \sum_{x\in\{0,1\}^n} |\alpha_x|^2 = 1$$

"Isolate the first qubit in ~~Prob[x]~~

$$|\psi\rangle = |0\rangle \otimes \left(\sum_{y\in\{0,1\}^{n-1}} \alpha_{0y}|y\rangle\right)$$

$$+ |1\rangle \otimes \left(\sum_{y\in\{0,1\}^{n-1}} \alpha_{1y}|y\rangle\right)$$

$$\cancel{\text{Prob}[0] = \langle\psi|P_0\otimes 1\otimes\cdots\otimes 1|\psi\rangle}$$

$$= \langle\psi|\left(|0\rangle\otimes\left(\sum_y \alpha_{0y}|y\rangle\right)\right)$$

$$= \left(\langle 0|\otimes\left(\sum_y \alpha_{0y}^*\langle y|\right) + \langle 1|\otimes\left(\sum_y \alpha_{1y}^*\langle y|\right)\right)$$

$$|0\rangle\otimes\left(\sum_y \alpha_{0y}|y\rangle\right)$$

$$= \underbrace{\langle 0|0\rangle}_{1} \otimes \sum_{y,y'} \alpha_{0y}^*\alpha_{0y'}\langle y|y'\rangle$$

$$+ \underbrace{\langle 1|0\rangle}_{0} \otimes \sum_{y,y'} \alpha_{1y}^*\alpha_{0y'}\langle y|y'\rangle$$

$$= \sum_{y,y'} \alpha_{0y}^*\alpha_{0y'}\underbrace{\langle y|y'\rangle}_{\delta_{yy'}}$$

$$= \sum_y \alpha_{0y}^*\alpha_{0y} = \sum_y |\alpha_{0y}|^2$$

$$= \text{Prob}[0]$$

Similarly, $\text{Prob}[1] = \sum_y |\alpha_{1y}|^2$

$$|\psi_0\rangle = \frac{\sum_y \alpha_{0y}|y\rangle}{\left\|\sum_y \alpha_{0y}|y\rangle\right\|} = \frac{\sum_y \alpha_{0y}|y\rangle}{\sqrt{\text{Prob}[0]}}$$

$$|\psi_1\rangle = \frac{\sum_y \alpha_{1y}|y\rangle}{\ldots}$$

Def: $A \in \mathbb{C}^{n\times n}$. $\overset{\neq 0}{v} \in \mathbb{C}^n$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$ if

$$Av = \lambda v$$
$$(A-\lambda I)v = 0$$

The spectrum of $A$ is the multiset of all eigenvalues of $A$ = multi-set of all roots $\lambda$ of the equation

$$\underbrace{\det(A-\lambda I)}_{n^{th}\text{ degree polynomial in }\lambda} = 0$$

denoted $\mathrm{char}_A(\lambda)$. (characteristic polynomial of $A$),

Def: $A$ is diagonal if

$[A]_{ij} = 0$ for all $i \neq j$

write $\mathrm{diag}(z_1, \dots, z_n)$

for $\begin{bmatrix} z_1 & & & 0 \\ & z_2 & & \\ & & \ddots & \\ 0 & & & z_n \end{bmatrix}$

Diag matrices are easy to work with: $D_1, D_2$ are $n\times n$ diag matrices

$D_1 = \mathrm{diag}(y_1, \dots, y_n)$

$D_2 = \mathrm{diag}(z_1, \dots, z_n)$,

then

$D_1 + D_2 = \mathrm{diag}(y_1+z_1, \dots, y_n+z_n)$

$D_1 D_2 = \mathrm{diag}(y_1 z_1, \dots, y_n z_n)$

In particular $D_1 D_2 = D_2 D_1$.

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Def: $A \in \mathbb{C}^{n\times n}$ is normal if $AA^* = A^*A$.

Thm (Spectral theorem): Every normal $A$ is unitarily conjugate to a diagonal matrix; $\exists$ diagonal $D = \mathrm{diag}(z_1, \dots, z_n)$, $\exists$ unitary $U$,

$$A = UDU^*$$

Further $\{z_1, \dots, z_n\}$ is the spectrum (set of eigenvalues) of $A$.

Equiv. Every normal $A$ has an eigenbasis (orthonormal basis of eigenvectors of $A$).

Consider $\{\underbrace{Ue_1}_{v_1}, \dots, \underbrace{Ue_n}_{v_n}\}$

Then $\{v_1, \dots, v_n\}$ is an orth. basis, and $Av_i = AUe_i \quad \forall i$

$= UDU^*Ue_i$

$= UDe_i = U(z_i e_i)$

$= z_i Ue_i = z_i v_i$

$\therefore$ $v_i$ is an eigenvector of $A$ with eigenval $z_i$.

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Lemma: $M \in \mathbb{C}^{n\times n}$ normal.

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$A$ is $k\times k$
$D$ is $(n-k)\times(n-k)$
$B$ is $k\times(n-k)$
$C$ is $(n-k)\times k$

(some $k$, $1 \leq k < n$)

If $C=0$, then $B=0$ and $A, D$ are both normal.

Proof: Assume $C=0$

$$MM^* = \left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right]\left[\begin{array}{c|c} A^* & 0 \\ \hline B^* & D^* \end{array}\right]$$

$$= \left[\begin{array}{c|c} AA^* + BB^* & BD^* \\ \hline DB^* & DD^* \end{array}\right]$$

$$\underset{M\text{ normal}}{=} M^*M = \left[\begin{array}{c|c} A^* & 0 \\ \hline B^* & D^* \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right]$$

$$= \left[\begin{array}{c|c} A^*A & A^*B \\ \hline B^*A & B^*B + D^*D \end{array}\right]$$

$A^*A = AA^* + BB^*$
$DD^* = B^*B + D^*D$

$BD^* = A^*B$
$DB^* = B^*A$

$A^*A = AA^* + BB^*$ $\quad$ $BD^* = A^*B$

$DD^* = B^*B + D^*D$ $\quad$ $DB^* = B^*A$

$$\overbrace{\mathrm{Tr}(A^*A) = \mathrm{Tr}(AA^*)}^{=} + \mathrm{Tr}(BB^*)$$

$$\begin{aligned} 0 &= \mathrm{Tr}(BB^*) \\ &= \mathrm{Tr}(B^{**}B^*) \\ &= \langle B^*, B^* \rangle = 0 \end{aligned}$$

$\therefore B^* = 0 \quad \therefore B = 0$

From $DD^* = B^*B + D^*D$
$= D^*D$

$\therefore D$ is normal.

Also,
$A^*A = AA^* + BB^* = AA^*$

$\therefore A$ is normal. //

Proof Sketch of the spectral theorm: (induction on $n$)

$A$ normal $n \times n$

$n = 1$ — trivial

Assume $n > 1$.

$\mathrm{char}_A(\lambda)$ is an $n^{th}$ degree polynomial over $\mathbb{C}$

So let $\lambda_1$ be a root of $\mathrm{char}_A(\lambda)$ [$\lambda_1$ exists b/c $\mathbb{C}$ is alg closed]

So $\lambda_1$ is an eigenvalue of $A$; let $b_1$ be a corresp eigenvector ($\|b_1\| = 1$ wlog)

Pick an orth. basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ arbitrarily such that 1st element is $b_1$

$A$ w.r.t. basis $\mathcal{B}$ has the form:

by Lemma

$$\left[\begin{array}{c|ccc} \lambda_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & (n-1)\times(n-1) & \end{array}\right]$$

$A'$ is normal so has an eigenbasis by inductive hyp. add $b_1$ to this basis to gen eigenbasis of $A$. //