Quantum gates, some identities

Correction

R_{ZZ}-gate $\neq$ C-Z

$$R_{ZZ}(\theta) = e^{-\frac{i\theta}{2}(Z\otimes Z)} = \left(\cos\frac{\theta}{2}\right) I\otimes I - \left(i\sin\frac{\theta}{2}\right) Z\otimes Z$$

$$= \begin{bmatrix} e^{-i\theta/2} & & & \\ & e^{i\theta/2} & & 0 \\ & & e^{i\theta/2} & \\ 0 & & & e^{-i\theta/2} \end{bmatrix}$$

$$C\text{-}Z = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$$

Controlled gates (cont)

U unitary (on n qubits $(n \geq 1)$)

C-U is an $(n+1)$-qubit unitary:

For any $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ (n-qubit state)

$$C\text{-}U(|0\rangle \otimes |\psi\rangle) = |0\rangle \otimes |\psi\rangle$$

$$C\text{-}U(\underset{\uparrow\ \text{control}}{|1\rangle} \otimes |\psi\rangle) = |1\rangle \otimes (U|\psi\rangle)$$

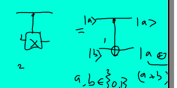
$$C\text{-}U = P_0 \otimes I + P_1 \otimes U$$

$$\begin{cases} P_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ P_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

$$C\text{-}U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes U = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & U \end{array}\right]$$

$$C\text{-}X = C\text{-}NOT = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & X \end{array}\right]$$

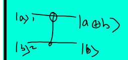
$$= \begin{array}{ccccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle & \\ \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right] & & & & \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

= 

$a, b \in \{0,1\}$ $(a+b)$ " $a \times$

This is a classical gate, that is, it maps basis states to basis state

If target comes first:

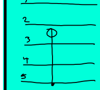


$$= I \otimes P_0 + X \otimes P_1$$

$$= \left[\begin{array}{c|c} P_0 & P_1 \\ \hline P_1 & P_0 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

($I \in \mathbb{C}^{2\times 2}$ identity)



$$= I \otimes I \otimes I \otimes I \otimes P_0 + I \otimes X \otimes I \otimes I \otimes P_1$$

$$= C\text{-}X_{5,2}$$

n-qubit register, U 1-qubit gate,

C-$U_{i,j}$ is controlled-U gate with control qubit i (i & target qubit j

Prop: ~~Every~~ Every n-qubit unitary operat… can be implemented exactly by an n-qubit circuit which only uses 1-qubit gates a… C-X gate,

[Caveat: # gates needed is exponential in n in the ~~worst~~ case]

Prop: ~~Every~~ ~~unitary~~ Every unitary gate on n … can be approximated to

If $\mathcal{H} \cong \mathbb{C}^n$ $(n = \dim \mathcal{H})$

$\mathcal{J} \cong \mathbb{C}^m$ $(m = \dim \mathcal{J})$,

then $\mathcal{L}(\mathcal{H}, \mathcal{J}) \cong \mathbb{C}^{m \times n}$

$\mathcal{L}(\mathcal{H}, \mathcal{J})$ (or $\mathbb{C}^{m\times n}$)

is a $\mathbb{C}$-space:

<u>Def</u>: Given $A, B \in \mathbb{C}^{m\times n}$

define $\langle A, B \rangle := \mathrm{Tr}(A^* B)$.

Then $\langle \circ, \circ \rangle$ satisfies all the properties of the Hermitian inner product.

Why?

$$\langle A, B\rangle = \mathrm{Tr}(\overbrace{A^* B}^{\in \mathbb{C}^{n\times n}})$$

$$= \sum_{i=1}^{n} [A^* B]_{ii}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} [A^*]_{ij} [B]_{ji}$$

$$= \sum_{i,j} ([A]_{ji})^* [B]_{ji}$$

= usual standard inner product of vectors arranged in $m \times n$ grids. So all properties of the inner prod on vectors.

More on 1-qubit operators & Bloch sphere

The set $\{I, X, Y, Z\}$ forms a basis for $\mathbb{C}^{2\times 2} \cong \mathcal{L}(\mathbb{C}^2)$

$\underbrace{\langle I, I\rangle}_{\text{not unit norm}} = \mathrm{Tr}(I^* I) = \mathrm{Tr}\, I = 2$

same with X, Y, Z

$\|I\| = \|X\| = \|Y\| = \|Z\| = \sqrt{2}$

all are orthogonal:

$$\langle X, Y\rangle = \mathrm{Tr}(X^* Y) = T(XY)$$

$$= \mathrm{Tr}(YX) = \mathrm{Tr}(-XY)$$

$$= -\mathrm{Tr}(XY) = -\langle X, Y\rangle$$

$\therefore \langle X, Y \rangle = 0$

[Same with any other two Pauli ops]

Alternately,

$$\mathrm{Tr}(XY) = \mathrm{Tr}(iZ) = i\,\mathrm{Tr}(Z)$$

$$= i\,\mathrm{Tr}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0$$

In particular,

$\mathrm{Tr}(\underset{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}{X}) = \langle I, X\rangle = 0$ (similarly, $\mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$)

Let $a = (a_x, a_y, a_z)$ be a point on the Bloch sphere.

So, $a_x, a_y, a_z \in \mathbb{R}$ and

$$a_x^2 + a_y^2 + a_z^2 = 1$$

Define $\sigma_a = a_x X + a_y Y + a_z Z$

σ_a is Hermitian and

$\sigma_a^2 = I$:

$$(a_x X + a_y Y + a_z Z)(a_x X + a_y Y + a_z Z)$$

$$= (a_x^2 + a_y^2 + a_z^2) I = I$$

[check that all other terms cancel]

So σ_a is also unitary:

$$\sigma_a = \sigma_a^* = \sigma_a^{-1}$$

σ_a corresponds to a π-rotation about the axis that goes through the origin and a.

Can define (in analogy to $R_x(\theta), R_y(\theta), R_z(\theta)$)

$$\boxed{R_a(\theta) = e^{-i\frac{\theta}{2}\sigma_a}}$$

every 1-qubit unitary is of this form, ignoring phase

$$R_a(\pi) = \left(\cos\frac{\pi}{2}\right) I - i\left(\sin\frac{\pi}{2}\right)\sigma_a$$

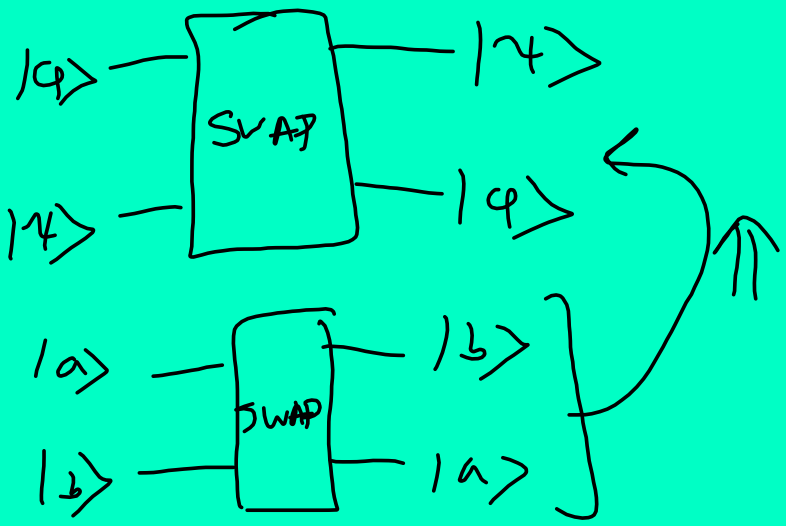
$$= -i\sigma_a \propto \sigma_a$$

S^3 parameterization of $\mathbb{C}^{2\times 2}$:

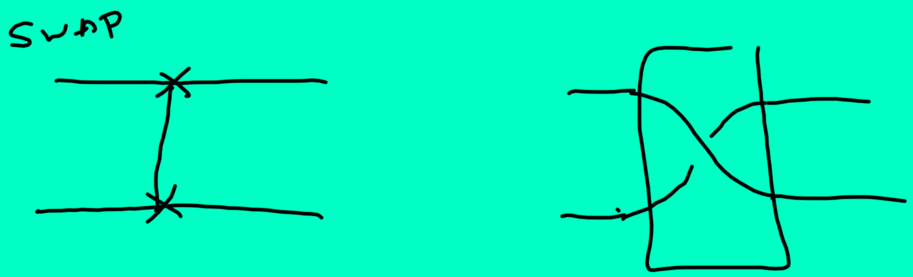
Suppose a has spherical coords θ, φ. Then for any $\omega \in \mathbb{R}$,

$$R_a(\omega) = \underbrace{R_z(\varphi) R_y(\theta)}_{\text{rotate north pole back to } a} \overbrace{R_z(\omega)}^{\text{rotate around north pole}} \underbrace{R_y(-\theta) R_z(-\varphi)}_{\text{move } a \text{ to north pole}}$$

2-qubit SWAP operator:



$a, b \in \{0,1\}$



$$\mathrm{SWAP}(|ab\rangle) = |ba\rangle$$

00	00
01	10
10	01
11	11

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$