$P = P^* P \iff P$ is an orthogonal projector

<u>Quantum circuit basics</u>:
1-qubit gates
controlled gates
measurement gates
} Today

1-qubit gates are unitary operators in $\mathcal{L}(\mathbb{C}^2)$
(2x2 unitary matrices)

Recall that 1-qubit states correspond to points on the Bloch sphere.

<u>Fact</u>: 1-qubit unitaries correspond to rigid rotations of the Bloch sphere. (up to phase factors)

<u>Notation</u>: Let A & B be any matrices (vector, operators) of the same type. Say that

$$A \propto B \quad (A \text{ is proportional to } B)$$

if $\exists \theta \in \mathbb{R}$,

$$A = e^{i\theta} B$$

<u>Fact</u>: A & B are 1-qubit unitaries, then A & B corresp. to the same Bloch sphere rotation iff $A \propto B$.

<u>Prop</u>: Let $A \in \mathbb{C}^{n \times n}$ such that $A^2 = I$, then $\forall \theta \in \mathbb{R}$,

$$e^{iA\theta} = (\cos\theta) I + i(\sin\theta) A$$

<u>Proof</u> (ex.)

Pauli operators I, X, Y, Z

$$I^2 = X^2 = Y^2 = Z^2 = I$$

<u>Also</u>:

$$\begin{cases} XY = -YX = iZ \\ YZ = -ZY = iX \\ ZX = -XZ = iY \end{cases}$$

$X \to Y \to Z \to X$

<u>Prop</u>: If $A \in \mathbb{C}^{n \times n}$ is Hermitian, then e^{iA} is unitary.

<u>Proof</u>:

$$e^{iA}(e^{iA})^* = e^{iA}\left(\sum_{j=0}^{\infty} \frac{(iA)^j}{j!}\right)^*$$

$$= e^{iA}\left(\sum_{j=0}^{\infty} \frac{((iA)^*)^j}{j!}\right)$$

$$= e^{iA}\left(\sum_{j=0}^{\infty} \frac{(i^* A^*)^j}{j!}\right)$$

$$= e^{iA}\left(\sum_{j=0}^{\infty} \frac{(-iA^*)^j}{j!}\right)$$

$$= e^{iA} e^{-iA^*} = e^{iA}e^{-iA} = e^{iA - iA} = e^0 = I \quad //$$

$A = A^*$ by assumption

X, Y, Z are Hermitian, so $e^{iX\theta}, e^{iY\theta}, e^{iZ\theta}$ are all unitary for all $\theta \in \mathbb{R}$.

<u>Def</u>: $\forall \theta \in \mathbb{R}$,

$$\left.\begin{aligned} R_x(\theta) &= e^{-iX\theta/2} \\ R_y(\theta) &= e^{-iY\theta/2} \\ R_z(\theta) &= e^{-iZ\theta/2} \end{aligned}\right\}$$

counter-clockwise rotations of the Bloch sphere about the x-, y-, z- axes, respectively

Consider $R_z(\varphi)$ $(\varphi \in \mathbb{R})$

$$R_z(\varphi) = e^{-iZ\varphi/2}$$

$$= \cos\left(-\frac{\varphi}{2}\right) \underbrace{I}_{\begin{pmatrix}1&0\\0&1\end{pmatrix}} + i\sin\left(-\frac{\varphi}{2}\right) \underbrace{Z}_{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}$$

$$= \begin{bmatrix} \cos\frac{\varphi}{2} - i\sin\left(\frac{\varphi}{2}\right) & 0 \\ 0 & \cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{bmatrix}$$

(next page)

$$\begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{bmatrix} \quad \left(\times\, e^{i\varphi/2}\right)$$

$$\propto \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$$

$$R_z(\varphi)|0\rangle \propto \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$R_z(\varphi)|1\rangle = R_z(\varphi)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\propto \begin{bmatrix} 0 \\ e^{i\varphi} \end{bmatrix} \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

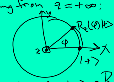
$$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$R_z(\varphi)|+\rangle$$

$$= \frac{1}{\sqrt{2}} R_z(\varphi)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \propto \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{i\varphi} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)$$

Looking from $z = +\infty$:



In particular, $\varphi = \frac{\pi}{2}$:

$\cancel{|x\rangle} := |+y\rangle := R_z\left(\frac{\pi}{2}\right)|+\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$

$(i = e^{i\pi/2})$

$$R_z(\pi)|+\rangle \propto |-\rangle$$

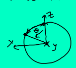
<u>Fact</u>: For every 1-qubit unitary U, there exist $\theta \in [0,\pi]$, $\varphi, \psi \in [0, 2\pi)$ such that

$$U \propto R_z(\varphi) R_y(\theta) R_z(\psi)$$

Geometrical explanation:

1st — $R_z(\psi)$ fixes North Pole

2nd — $R_y(\theta)$ moves the North Pole in the x,z plane toward the +X axis to any desired latitude.



from $y = +\infty$

3rd — $R_z(\varphi)$ furth moves the point to any desired longitude (keeping the same latitude)

Summary 2nd & 3rd rotations move the north pole to any desired point on the sphere (with spherical coords θ, φ).

1st rotation $R_z(\psi)$ "prerotates" fixing the north pole, providing the extra degree of freedom for the rotation.

φ, θ, ψ are <u>Euler angles</u>

& they parameterization of all rotations in $\mathbb{R}^3$ fixing the origin.

[quaternions]

Important 1-qubit gates:

Pauli gates

$$H := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ <u>Hadamard gate</u>

$$S := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$ <u>Phase gate</u>

$$T := \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix}$$

Geometry:

$$X \propto R_x(\pi)$$
$$Y \propto R_y(\pi)$$
$$Z \propto R_z(\pi)$$

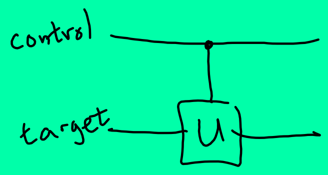
$$S \propto R_z(\pi/2) \quad [S^2 = Z]$$

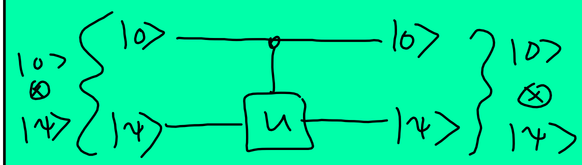
$$T \propto R_z\left(\frac{\pi}{4}\right) \quad [T^2 = S]$$

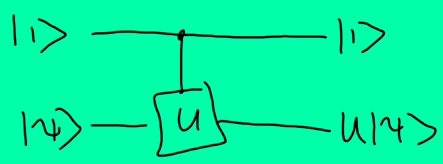
Controlled gates.

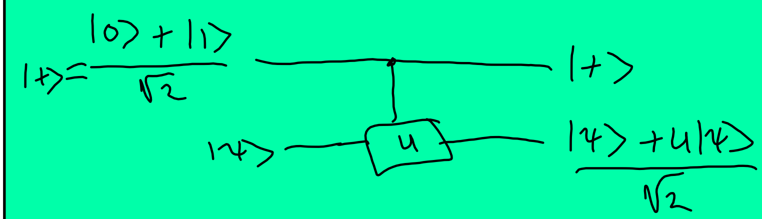
Let U by a unitary (1-qubit, say).

C-U is a 2-qubit gate



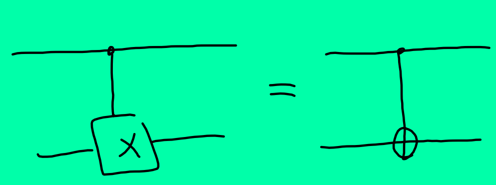






C-U is linear & unitary

Ex: C-X (C-NOT)



R_{zz}