Combined systems (cont.)

Product bases
Inner products ⟵ Product states
Multiqubit registers
Quantum circuit model

Def: Let $\mathcal{H}$ & $\mathcal{J}$ be $\mathbb{C}$-spaces, $\{u_1, \dots, u_m\}$ basis for $\mathcal{H}$, $\{v_1, \dots, v_n\}$ basis for $\mathcal{J}$. The product of these two bases is the set

$$\{u_i \otimes v_j : 1 \le i \le m,\ 1 \le j \le n\} \subseteq \mathcal{H} \otimes \mathcal{J}$$

Prop: This set is an (orthonormal) basis for $\mathcal{H} \otimes \mathcal{J}$.

Lemma: $u_1, v_1 \in \mathcal{H}$. $u_2, v_2 \in \mathcal{J}$.

$$\langle \underbrace{u_1 \otimes v_1}_{\in \mathcal{H}\otimes\mathcal{J}}, \underbrace{u_2 \otimes v_2}_{\in \mathcal{H}\otimes\mathcal{J}} \rangle = \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle.$$

Proof:

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle = (u_1 \otimes v_1)^* (u_2 \otimes v_2)$$
$$= (u_1^* \otimes v_1^*)(u_2 \otimes v_2)$$
$$= u_1^* u_2 \otimes v_1^* v_2$$
$$= \langle u_1, u_2 \rangle \otimes \langle v_1, v_2 \rangle \quad \Big\} \otimes \text{ of } 1\times 1 \text{ matrices}$$
$$= \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle. \quad //$$

Proof of the prop:

For any $\underbrace{u_i, u_j}_{\text{basis of } \mathcal{H}}, \underbrace{v_k, v_l}_{\text{basis of } \mathcal{J}}$

$$\langle u_i \otimes v_k, u_j \otimes v_l \rangle$$
$$= \langle u_i, u_j \rangle \langle v_k, v_l \rangle \quad \text{(by Lemma)}$$
$$= \delta_{ij}\, \delta_{kl} = \delta_{(i,k)(j,l)}$$
$$= \begin{cases} 1 & \text{if } i=j \ \&\ k=l \\ 0 & \text{o.w.} \end{cases}$$
$$= \begin{cases} 1 & \text{if } (i,k)=(j,l) \\ 0 & \text{o.w.} \end{cases}$$

So product is an orthon. basis. //

Ex: Standard basis
$e_1, \dots, e_m$ for $\mathcal{H}$
$e_1, \dots, e_n$ for $\mathcal{J}$

$$e_{(i,j)} = e_i \otimes e_j = e_{(i-1)n+j} = \begin{bmatrix} 0 \\ \hline e_j \\ \hline 0 \\ \hline 0 \end{bmatrix}$$

product is the standard basis for the tensor product.

$$\mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathcal{H}_n = \mathbb{C}^2$$

Each $\mathcal{H}_j$ is a 1-qubit $\mathbb{C}$-space.

$\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is a n-qubit $\mathbb{C}$-space ("n-qubit register").

$\{\overset{e_1}{|0\rangle}, \overset{e_2}{|1\rangle}\}$ std basis for each $\mathcal{H}_j$.

Standard basis (computational basis) for n-qubit register is

$$\{\underbrace{|b_1\rangle \otimes \cdots \otimes |b_n\rangle}_{\underbrace{|b_1\rangle|b_2\rangle \cdots |b_n\rangle}_{|b_1 \cdots b_n\rangle}} : b_1, \dots, b_n \in \{0,1\}\}$$

Neglected part of the proof of the prop:

$\{u_i \otimes v_j : 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $\mathcal{H} \otimes \mathcal{J}$.

(proof above only shows it is an orthnormal set).

Product is linearly indep by a HW1 exercise, but there are mn many vectors, & this equals $\dim(\mathcal{H} \otimes \mathcal{J})$.

$\therefore$ product spans $\mathcal{H} \otimes \mathcal{J}$.

$\therefore$ product is a basis for $\mathcal{H} \otimes \mathcal{J}$. //

n-qubit register $(\mathbb{C}^2)^{\otimes n}$ has computational basis

$$\{|x\rangle : x \in \{0,1\}^n\}$$

Ex: $n=2$. $(\mathbb{C}^2)^{\otimes 2} \cong \mathbb{C}^4$

basis $\{\underset{e_1}{|00\rangle}, \underset{e_2}{|01\rangle}, \underset{e_3}{|10\rangle}, \underset{e_4}{|11\rangle}\}$

$$|00\rangle = e_1 \otimes e_1 = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot 1\\1\cdot 0\\0\cdot 1\\0\cdot 0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = e_1$$

etc.

Generally $\mathcal{H}\otimes\mathcal{J}$ is spanned by vectors of the form $u\otimes v$ ($u\in\mathcal{H}$, $v\in\mathcal{J}$) but not equal to it.

Ex: $\mathcal{H} = \mathcal{J} = \mathbb{C}^2$

$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is not the tensor product of 2 1-qubit states $|\varphi\rangle\otimes|\psi\rangle$ (any $|\varphi\rangle, |\psi\rangle \in \mathbb{C}^2$)

The qubits in this state are entangled.

In physics, can combine any 2 systems: $\mathcal{H}$, $\mathcal{J}$ to get $\mathcal{H}\otimes\mathcal{J}$.

If $|\varphi\rangle\in\mathcal{H}$ is the state of $\mathcal{H}$ & $|\psi\rangle\in\mathcal{J}$ is the state of $\mathcal{J}$, then $|\varphi\rangle\otimes|\psi\rangle$ is the state of $\mathcal{H}\otimes\mathcal{J}$.

Evolving $\mathcal{H}$ by some unitary $U\in\mathcal{L}(\mathcal{H})$ results in the state $(U|\varphi\rangle)\otimes|\psi\rangle$ (if nothing is done with $\mathcal{J}$)

$$(U|\varphi\rangle)\otimes|\psi\rangle = \underbrace{(U\otimes I)}_{\mathcal{L}(\mathcal{H}\otimes\mathcal{J}) \text{ unitary}}(|\varphi\rangle\otimes|\psi\rangle)$$

More generally, if $\mathcal{J}$ also evolves indep of $\mathcal{H}$ by a unitary V, then the resulting state is

$$(U|\varphi\rangle)\otimes(V|\psi\rangle) = \underset{\in\mathcal{L}(\mathcal{H}\otimes\mathcal{J}) \text{ unitary}}{(U\otimes V)}(|\varphi\rangle\otimes|\psi\rangle)$$

If state of $\mathcal{H}\otimes\mathcal{J}$ is not of the form $|\varphi\rangle\otimes|\psi\rangle$ the two systems are entangled.

Prop: $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is an entangled state

Proof: Let $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\psi\rangle = \gamma|0\rangle + \delta|1\rangle$
some $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ ($|\alpha|^2+|\beta|^2 = |\gamma|^2+|\delta|^2=1$)
arbitrary 1-qubit states

$$|\varphi\rangle\otimes|\psi\rangle = (\alpha|0\rangle+\beta|1\rangle)\otimes(\gamma|0\rangle+\delta|1\rangle)$$

$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

Suppose this state equals $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$. Then

$$\alpha\gamma = \tfrac{1}{\sqrt{2}} = \beta\delta,$$
$$\alpha\delta = \beta\gamma = 0 \quad \lightning$$

so $|\Phi^+\rangle$ is entangled.

Def:

$$|\Phi^+\rangle := \frac{|00\rangle+|11\rangle}{\sqrt{2}}$$
$$|\Phi^-\rangle := \frac{|00\rangle-|11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle := \frac{|01\rangle+|10\rangle}{\sqrt{2}}$$
$$|\Psi^-\rangle := \frac{|01\rangle-|10\rangle}{\sqrt{2}}$$

These are all maximally entangled states of 2 qubits, called Bell states. The form an orthonormal basis for $\mathbb{C}^4$ (Bell basis).

Quantum circuit model

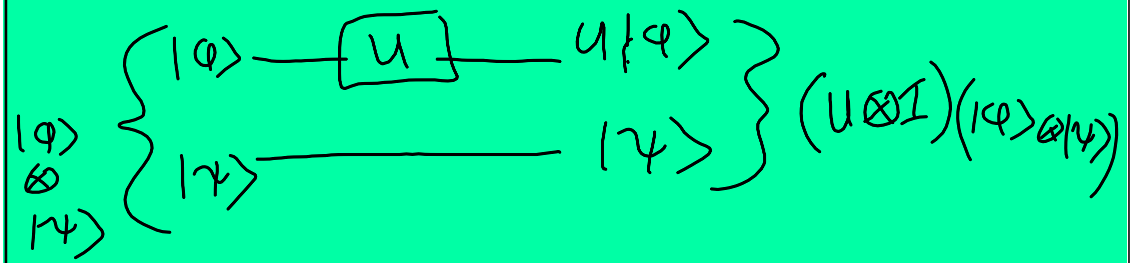
n-qubit register evolving via unitary ops applied to small subsets of the n-qubits.

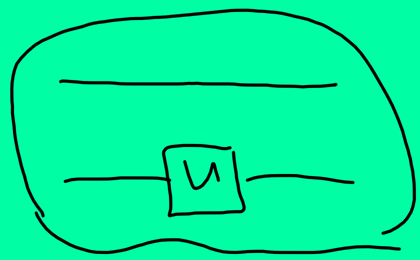
Each qubit represented by a horizontal line ("wire"), Unitary operator are called gates are superimposed over the wires. ("time") Sequence of gate application goes from left to right

$|\varphi\rangle$ ———— $|\varphi\rangle$

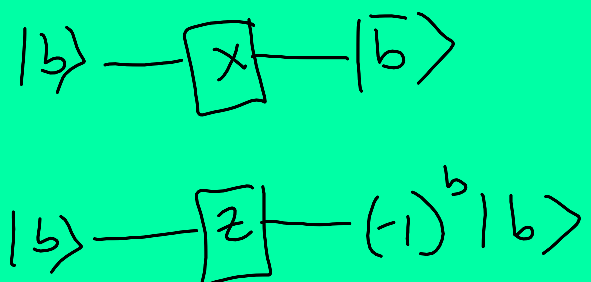
$|\varphi\rangle$ —$\boxed{U}$— $U|\varphi\rangle$ $\quad$ $U\in\mathcal{L}(\mathbb{C}^2)$ unitary

$|\varphi\rangle$ —$\boxed{U}$—$\boxed{V}$— $VU|\varphi\rangle$ $\quad$ $V\in\mathcal{L}(\mathbb{C}^2)$ unitary





$(I \otimes U)$



$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

$$Y = iXZ$$