- Isolated systems & unitary evolution
- Combined systems & the tensor product

Change of (orthonormal) basis

<u>Def</u>: $\mathcal{H}, \mathcal{J}$ $\mathbb{C}$-spaces

$\mathcal{L}(\mathcal{H}, \mathcal{J})$ is space of all ~~operators~~ linear maps $\mathcal{H} \to \mathcal{J}$

$\dim(\mathcal{H}) = n$ | $\mathcal{H} \cong \mathbb{C}^n$
$\dim(\mathcal{J}) = m$ | $\mathcal{J} \cong \mathbb{C}^m$

Then $\mathcal{L}(\mathcal{H}, \mathcal{J}) \cong \mathbb{C}^{m \times n}$
$\cong \mathbb{C}^{mn}$

$\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$
"operators on $\mathcal{H}$"

Given a (orth. by ~~dotan~~ $\mathcal{H}$) basis for $\mathcal{H}$, say $\{u_1, \dots, u_n\}$, there is a unique $U \in \mathcal{L}(\mathcal{H})$ such that $u_j = Ue_j$ $(\forall j)$

U preserves inner products:

$$\langle Ue_j, Ue_k \rangle = \langle u_j, u_k \rangle$$

$$\begin{array}{c|l} \langle Ue_j, Ue_k \rangle & = \delta_{jk} \\ \parallel & \\ \langle U^*U e_j, e_k \rangle & = \langle e_j, e_k \rangle \\ \parallel & \\ \langle e_j, \underline{U^*U} e_k \rangle & = \langle e_j, I e_k \rangle \end{array}$$

Now: Any $A \in \mathcal{L}(\mathcal{H})$

$$[A]_{jk} = e_j^* A e_k$$

$$= \begin{array}{c} \hphantom{[0 - 0\ } j \hphantom{\ 0 - 0]} \\ \begin{bmatrix} 0 & - & 0 & 1 & 0 & - & 0 \end{bmatrix} \end{array} A \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow k \\ \\ \\ \\ \end{array}$$

$$= \langle e_j, Ae_k \rangle$$

A determined completely by inner prods $\langle e_j, Ae_k \rangle$ $\forall j,k$, so $U^*U = I$. $[U^* = U^{-1}]$

Given $A \in \mathbb{C}^{n \times n}$

express A with respect to basis $\{u_1, \dots, u_n\}$:

by a matrix A' such that

$$[A']_{jk} := \langle u_j, Au_k \rangle$$

defines a new matrix A': $\forall j,k$

$$[A']_{jk} = \langle u_j, Au_k \rangle$$
$$= \langle Ue_j, AUe_k \rangle$$
$$= \langle e_j, U^*AUe_k \rangle$$
$$= [U^*AU]_{jk}$$

$\therefore A' = U^*AU$

$\therefore UA' = AU$

$\therefore UA'U^* = A$

A & A' are <u>unitarily conjugate</u>.

(Almost) all important properties of operators are invariant under unitary conjugation.

<u>Prop</u>: For any $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$,

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

<u>Cor</u>: $A, B, C, D, \dots, F$

$$\mathrm{Tr}(ABCD \cdots F) = \mathrm{Tr}(FABCD \cdots)$$
"cyclic property of Tr"

<u>Cor</u>: $\mathrm{Tr}(UAU^*)$ $\left[\begin{array}{l} U \text{ unitary} \\ U, A \in \mathcal{L}(\mathcal{H}) \end{array}\right]$

(1)
$$\mathrm{Tr}(UAU^*) = \mathrm{Tr}(U^*UA) = \mathrm{Tr}(A)$$

Tr is invariant under unitary conj.

(2) $(UAU^*)^* = U^{**}A^*U^* = UA^*U^*$

"adjoint of conj is conj of the adjoint"

(3) A is Hermitian iff UAU^* is Hermitian.

Etc.

<u>Isolated system evolution</u>

Given a closed, bounded physical system $\mathcal{H}$ ($\dim \mathcal{H} = n$)

The state of $\mathcal{H}$ evolves in time according to a (time-dependent) unitary operator:

For any times $t_1, t_2 \in \mathbb{R}$ There is a unitary operator U_{t_2,t_1} (depending on t_1, t_2) such that, for any state of $\mathcal{H}$ at time t_1, say $|\psi\rangle_{t_1}$, the state of $\mathcal{H}$ at time t_2 is $U_{t_2,t_1}|\psi\rangle_{t_1} =: |\psi\rangle_{t_2}$

Crucially, U_{t_2,t_1} does not depend on the state of the system at any time, only on the system itself (i.e., internal forces)

<u>Furthermore</u>: $\forall t_1, t_2, t_3 \in \mathbb{R}$,

1.) $U_{t_3,t_1} = U_{t_3,t_2} U_{t_2,t_1}$

2.) $U_{t_1,t_1} = I$

3.) $U_{t_1,t_2} = U_{t_2,t_1}^{-1} = U_{t_2,t_1}^*$

(not assuming $t_2 \geq t_1$).

Fix time $t_0 = 0$, & let

$$U_t := U_{t,0} \quad \forall t$$

$U_0 = I$

$U_{-t} = U_t^*$

$U_{t_1+t_2} = U_{t_1}U_{t_2} = U_{t_2}U_{t_1}$

Schrödinger's Equation:

$$\boxed{\frac{dU_t}{dt} = -\frac{i}{\hbar} H_t U_t}$$

where $H_t \in \mathcal{L}(\mathcal{H})$ for all t, called the <u>Hamiltonian</u> of the system.

Common case: $H_t = H$ is time-independent. Then

$$U_t = e^{-iHt/\hbar}$$

[$\hbar$ is reduced Planck's constant in units of energy $\times$ time $= \frac{\text{energy}}{\text{angular frequency}}$]

Choose physical units so that $\hbar = 1$:

$$U_t = e^{-iHt}$$

A quantum circuit applies unitary operators to groups of qubits so they evolve in time:

$$\xrightarrow{\text{time}}$$

$$|\psi\rangle \;—\!\!—\; \boxed{U} \;—\!\!—\; U|\psi\rangle$$

1-qubit unitary operators correspond to rigid rotations of the Bloch sphere (more later).

Combined systems

Def: $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{r \times s}$ $\quad \forall m, n, r, s$

we define the tensor product (Kronecker product, outer product)

$A \otimes B \in \mathbb{C}^{mr \times ns}$ as

$$A \otimes B = \begin{bmatrix} [A]_{11} B & [A]_{12} B & \cdots \\ [A]_{21} B & [A]_{22} B & \\ \vdots & & \end{bmatrix}.$$

Fundamental properties:

1. $(A \otimes B)^* = A^* \otimes B^*$
2. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
3. $(A \otimes B)(C \otimes D) = AC \otimes BD$ (assuming AC & BD well-defined)
4. If $A \in \mathbb{C}^{m \times 1}$ (column vector), $B \in \mathbb{C}^{1 \times s}$ (row vector), then $A \otimes B = AB \in \mathbb{C}^{m \times s}$
5. $A \otimes [1] = [1] \otimes A = A$

Given $\mathbb{C}$-spaces $\mathcal{H}, \mathcal{J}$, define the tensor product of $\mathcal{H}$ and $\mathcal{J}$ as

$$\mathcal{H} \otimes \mathcal{J} := \operatorname{span}\{ \underbrace{u \otimes v}_{\text{column vectors}} ; u \in \mathcal{H}, v \in \mathcal{J} \}$$

$\dim(\mathcal{H} \otimes \mathcal{J}) = (\dim \mathcal{H})(\dim \mathcal{J})$.

$\mathcal{H} \otimes \mathcal{J}$ is the combined physical system of $\mathcal{H}$ together with $\mathcal{J}$.

Two more properties of $\otimes$:

$\forall A, B, C$ (B, C same size)

6. $A \otimes (B + C) = A \otimes B + A \otimes C$
   $(B + C) \otimes A = B \otimes A + C \otimes A$
7. $\forall a \in \mathbb{C}$,
   $(aA) \otimes B = A \otimes (aB) = a(A \otimes B)$.

"$\otimes$ is bilinear"