QM fundamentals:

<u>Projective Measurements</u>

"Axiom" A physical system corresponds to a Hilbert space $\mathcal{H}$ (If $\mathcal{H}$ is finite-dimentional, then the system is <u>bounded</u>).

At any given time, the system is in a <u>state</u> represented by a unit vector $|\psi\rangle \in \mathcal{H}$

$|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same physical state

[We will redefine a state later to fix this defect.]

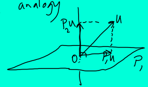
<u>Def</u>: Given a $\mathbb{C}$-space $\mathcal{H}$, A <u>complete set of orthogonal projectors</u> (<u>csop</u>) is a set $\{P_1, \dots, P_k\}$ of projectors on $\mathcal{H}$ such that

$$\sum_{j=1}^{k} P_j = I \quad (= id_{\mathcal{H}})$$

<u>Fact</u>: in a csop $\{P_1, \dots, P_k\}$,

$$P_i P_j = 0 \text{ for all } i \neq j.$$

<u>Ex</u>: $\mathbb{R}^3$ analogy



Given a system $\mathcal{H}$ (bounded), A <u>projective measurement</u> on $\mathcal{H}$ corresponds to a csop $\{P_1, \dots, P_k\}$ on $\mathcal{H}$.

Given $\mathcal{H}$ is in state $|\psi\rangle$, Measuring with this csop results in one of the "outcomes" $1, \dots, k$, where outcome i occurs with probability $\| P_i|\psi\rangle \|^2$

Given outcome i, the post-measurement state changes to $\dfrac{P_i|\psi\rangle}{\|P_i|\psi\rangle\|}$.

<u>Note</u>: $\| P_i|\psi\rangle \|^2$

$$= \left\langle P_i|\psi\rangle, P_i|\psi\rangle \right\rangle$$

$$= (P_i|\psi\rangle)^* P_i|\psi\rangle$$

$$= |\psi\rangle^* P_i^* P_i|\psi\rangle$$

$$= \langle\psi|P_i P_i|\psi\rangle$$

$$= \underline{\langle\psi|P_i|\psi\rangle} = Pr[i]$$

<u>Note</u>: $|\psi\rangle$ & $e^{i\theta}|\psi\rangle$ give the same prob dist of outcomes of a proj. measurement;

Let $|\psi'\rangle := e^{i\theta}|\psi\rangle$

$$\langle\psi'|P_i|\psi'\rangle$$

$$= (e^{i\theta}|\psi\rangle)^* P_i (e^{i\theta}|\psi\rangle)$$

$$= \langle\psi|e^{-i\theta} P_i e^{i\theta}|\psi\rangle$$

$$= \langle\psi|P_i|\psi\rangle$$

Post-measurement state is

$$\frac{P_i|\psi'\rangle}{\|P_i|\psi'\rangle\|} = \frac{P_i|\psi'\rangle}{\sqrt{Pr[i]}}$$

$$= e^{i\theta}\frac{P_i|\psi\rangle}{\sqrt{Pr[i]}} = e^{i\theta} \text{ time post-meas state with } |\psi\rangle$$

Ex: $\mathcal{H} = \mathbb{C}^2$

$$\left.\begin{array}{l} P_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ P_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array}\right\} \text{csop}$$

More generally, $\mathcal{H} = \mathbb{C}^n$

$P_1, \dots, P_n$ so that

$$P_j = \begin{array}{c} \\ j \rightarrow \end{array} \begin{array}{c} j \\ \downarrow \\ \begin{bmatrix} 0 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 0 \end{bmatrix} \end{array}$$

<u>Def</u>: A <u>qubit</u> is a 2-dimensional physical system. ($\mathcal{H} = \mathbb{C}^2$)

$$|0\rangle := e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (= |\uparrow\rangle)$$

$$|1\rangle := e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (= |\downarrow\rangle)$$

(qubit = "quantum bit")

1-qubit states. Generally a qubit state is of the form

$$|\psi\rangle := \alpha|0\rangle + \beta|1\rangle$$

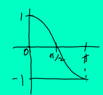
for some $\alpha, \beta \in \mathbb{C}$ such that

$$|\alpha|^2 + |\beta|^2 = 1$$

Can assume (by adjusting the phase factor) that $\alpha \geq 0$.

So $0 \leq \alpha \leq 1$

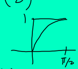
Choose unique θ, $0 \leq \theta \leq \pi$ such that $\alpha = \cos\left(\frac{\theta}{2}\right)$



$$|\alpha|^2 = \cos^2\left(\frac{\theta}{2}\right) = \alpha^2$$

$$|\beta|^2 = 1 - |\alpha|^2 = 1 - \cos^2\left(\frac{\theta}{2}\right) = \sin^2\left(\frac{\theta}{2}\right)$$

$$\therefore |\beta| = +\sin\left(\frac{\theta}{2}\right)$$



So, $\beta = e^{i\varphi} \sin\left(\frac{\theta}{2}\right)$

for some φ

(φ is unique if constrained to $[0, 2\pi)$.)

$$|\psi\rangle := \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \varphi < 2\pi$$

Bit value projective measurement:

$$P_{|0\rangle} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{|1\rangle} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

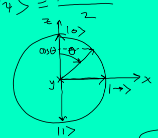
$$P_{|0\rangle}|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ 0 \end{bmatrix}$$

$$\langle\psi|P_{|0\rangle}|\psi\rangle = \begin{bmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ 0 \end{bmatrix}$$

$$= \cos^2\left(\frac{\theta}{2}\right) = \frac{1 + \cos\theta}{2}$$

Check:

$$\langle\psi|P_{|1\rangle}|\psi\rangle = \frac{1 - \cos\theta}{2}$$



$\theta = \frac{\pi}{2}$

$$|\rightarrow\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\varphi}\sin\frac{\pi}{4}|1\rangle$$

$[\varphi := 0 \text{ for } |\rightarrow\rangle]$

$\left[\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}\right.$

$$|\rightarrow\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} =: |+\rangle$$

$|\leftarrow\rangle$: $\varphi := \pi$:

$$= \frac{|0\rangle + e^{i\pi}|1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} =: |-\rangle$$

θ & φ are the spherical coordinates of the electron spin direction.

<u>Def</u>: The Pauli spin matrices are

$$\sigma_0 := I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_1 := X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 := Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 := Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Given $|\psi\rangle = \cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}$

Consider the operator

$$\rho := |\psi\rangle\langle\psi| \qquad \begin{bmatrix} \text{1-dimensional} \\ \text{projector} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{bmatrix}$$

$$\left[e^{i\varphi} = \cos\varphi + i\sin\varphi \right]$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & 1-\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\left(I + \sin\theta\cos\varphi\, X + \sin\theta\sin\varphi\, Y + \cos\theta\, Z \right)$$

Spherical $\longrightarrow$ cartesian:

$$x = \sin\theta\cos\varphi$$

$$y = \sin\theta\sin\varphi$$

$$z = \cos\theta$$