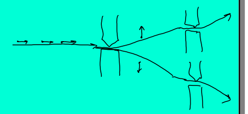
Stern-Gerlach (cont.)



<u>Matrices & operators</u>

$\mathbb{C}^n$ — space of all n-dim (column) vectors over $\mathbb{C}$

$\mathbb{C}^{m\times n}$ — all $m\times n$ matrices over $\mathbb{C}$

$A \in \mathbb{C}^{m\times n}$ corresponds to a linear map $\mathbb{C}^n \to \mathbb{C}^m$

by $v \in \mathbb{C}^n \longmapsto Av \in \mathbb{C}^m$

Recall: Given $A \in \mathbb{C}^{m\times n}$ define $A^* \in \mathbb{C}^{n\times m}$ (operator from $\mathbb{C}^m$ to $\mathbb{C}^n$)

as conjugate transpose of A.

<u>Prop</u>: Given any $v \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$

$$\underbrace{\langle u, Av\rangle}_{\text{inner prod in } \mathbb{C}^m} = \underbrace{\langle A^*u, v\rangle}_{\text{inner prod in } \mathbb{C}^n}.$$

Note: $A^{**} = A$

$\therefore \langle v, A^*u\rangle = \langle Av, u\rangle = \langle A^{**}v, u\rangle$

<u>Proof</u>:

$$\langle u, Av\rangle \overset{\text{last time}}{=} \underbrace{u^*Av} = \underbrace{(A^*u)^*}v = \langle A^*u, v\rangle$$

A matrix in $\mathbb{C}^{n\times n}$ is <u>square</u>.

<u>Def</u>: A square matrix A is <u>Hermitian</u> if $A = A^*$

<u>Hermitean</u> ("self-adjoint")

A is <u>unitary</u> if

$$AA^* = I \quad (\text{equiv: } A^*A = I)$$

i.e., $A^{-1} = A^*$

"Unitary" is the $\mathbb{C}$-space analog of <u>orthogonal</u> for real matrices.

<u>Prop</u>: If A is unitary, then $\langle Au, Av\rangle = \langle u, v\rangle$ for all $u, v \in \mathbb{C}^n$.

<u>Pf</u>: $\langle Au, Av\rangle = \langle A^*Au, v\rangle = \langle Iu, v\rangle = \langle u, v\rangle$ //

$\therefore$ unitary matrices correspond to inner product-preserving linear maps: Applying A to a unit vector is a unit vector. Applying A to each of 2 orthogonal vectors results in orthogonal vectors.

A matrix (operator) $P \in \mathbb{C}^{n\times n}$ is a <u>projector</u> (or <u>orthogonal projector</u>) if

1) $PP = P$ ("P is idempotent")

2) $P = P^*$ (P is Hermitian).

<u>Def</u>. Any $A \in \mathbb{C}^{m\times n}$,

$\text{img}(A) = \{Av : v \in \mathbb{C}^n\}$

("image of A") $\left(= A(\mathbb{C}^n)\right)$

$\text{img}(A)$ is a vector subspace of $\mathbb{C}^m$.

If P is a projector and $v \in \text{img}(P)$, then $Pv = v$. <u>Pf</u>: $v = Pu$ for some u

$Pv = PPu = Pu = v$ //

<u>Def</u>: $A \in \mathbb{C}^{m \times n}$. The <u>kernel of A</u> ($\ker(A)$) is $\ker(A) = \{u \in \mathbb{C}^n : Au = \underline{0}\}$

(vector subspace of $\mathbb{C}^n$)

<u>Def</u>: Given $S \subseteq \mathbb{C}^n$ vector subspace of $\mathbb{C}^n$, define the orthogonal complement of S to be

$$S^\perp := \{u \in \mathbb{C}^n : \langle u, v \rangle = 0 \text{ for all } v \in S\}$$

$S^\perp$ is a vector subspace.

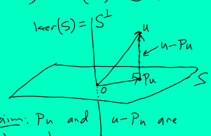
$$\left.\begin{aligned} S \cap S^\perp &= \{0\} \\ \mathbb{C}^n &= S + S^\perp \end{aligned}\right\} \quad \mathbb{C}^n = \underline{S \oplus S^\perp}$$

$\forall v \in \mathbb{C}^n$, $\exists$ unique $x \in S$, $y \in S^\perp$,

$$v = x + y.$$

$$n = \dim(S) + \dim(S^\perp).$$

<u>Prop</u>: Given any subspace $S \subseteq \mathbb{C}^n$, there exists a unique projector P with $S = \operatorname{img}(P)$. For such a P, $\ker(P) = S^\perp$.



<u>Claim</u>: Pu and $u - Pu$ are orthogonal:

$$\langle Pu, u - Pu \rangle = \langle Pu, u \rangle - \langle Pu, Pu \rangle$$
$$= \langle Pu, u \rangle - \langle P^* Pu, u \rangle$$
$$= \langle Pu, u \rangle - \langle PPu, u \rangle$$
$$= \langle Pu, u \rangle - \langle Pu, u \rangle = 0.$$

Electron spin states

<u>Dirac notation</u>:

Given $\mathbb{C}^n$, a unit column vector in $\mathbb{C}^n$ may be written as

$$|\psi\rangle \quad (\text{"ket vector"})$$

here, ψ is some label to identify the vector.

Write $|\psi\rangle^* = \langle\psi|$ ("bra vector")

For $|\varphi\rangle$, $|\psi\rangle$, their inner product is

$$\Big\langle |\varphi\rangle, |\psi\rangle \Big\rangle$$

$$= |\varphi\rangle^* |\psi\rangle = \langle\varphi|\psi\rangle$$

"bracket" (bra-ket)

Quantum theory of electron spin posits that the spin state of an electron is a unit vector in $\mathbb{C}^2$.

Standard basis

$$\{e_1, e_2\} = \{|\uparrow\rangle, |\downarrow\rangle\}$$

General spin state of an electron is of the form

$$|\psi\rangle := \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

for some $\alpha, \beta \in \mathbb{C}$.

$$1 = \langle\psi|\psi\rangle = \left(\alpha^* \langle\uparrow| + \beta^* \langle\downarrow|\right)\left(\alpha|\uparrow\rangle + \beta|\downarrow\rangle\right)$$

$$\langle\psi|\psi\rangle = |\psi\rangle^* |\psi\rangle$$

$$= \alpha^*\alpha \overbrace{\langle\uparrow|\uparrow\rangle}^{1} + \cancel{\alpha^*\beta\langle\uparrow|\downarrow\rangle} \rightarrow 0$$
$$0 \leftarrow + \cancel{\beta^*\alpha\langle\downarrow|\uparrow\rangle} + \beta^*\beta \underbrace{\langle\downarrow|\downarrow\rangle}_{1}$$

$$= \alpha^*\alpha + \beta^*\beta = \underbrace{|\alpha|^2}_{\geq 0} + \underbrace{|\beta|^2}_{\geq 0} = 1.$$

Given spin state $|\psi\rangle = \underline{\alpha|\uparrow\rangle} + \beta|\downarrow\rangle$

S-G apparatus will send spin up with prob $|\alpha|^2$

(& spin down with prob $|\beta|^2$)

α, β are <u>probability amplitudes</u>.

Take $|\cdot|^2$ to get the probabilities.

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

$$e^{i\theta}|\psi\rangle = e^{i\theta}\alpha|\uparrow\rangle + e^{i\theta}\beta|\downarrow\rangle.$$

$$\text{Prob}[\uparrow] = |e^{i\theta}\alpha|^2$$
$$= (e^{i\theta}\alpha)^*(e^{i\theta}\alpha)$$
$$= e^{-i\theta}\alpha^* e^{i\theta}\alpha$$
$$= \alpha^*\alpha = |\alpha|^2$$

Prob of $\uparrow$ for $|\psi\rangle$ same as for $e^{i\theta}|\psi\rangle$

$e^{i\theta}$ called a <u>global phase factor</u>.

$\therefore$ Vectors that differ by a phase factor correspond to the same physical state.