Math	Physics
Orth. bases	Electron spin
Adjoints	Stern-Gerlach experiment
Prob. dist's	

<u>Def</u>: Let $\mathcal{H}$ be a $\mathbb{C}$-space.

An <u>orthonormal</u> <u>set</u> is a set of vectors $u_1, \dots, u_k$ such that

$$\langle u_i, u_j \rangle = \underbrace{\delta_{ij}}_{\text{Kronecker Delta}} := \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$\therefore$ each u_i is a unit vector:

$\sqrt{\langle u_i, u_i \rangle} = \|u_i\| = \delta_{ii} = 1$ (u_i is <u>normal</u>)

& u_i & u_j are <u>orthogonal</u> (i.e. $\langle u_i, u_j \rangle = 0$) for $i \neq j$.

An <u>orthonormal</u> <u>basis</u> is an orthonormal set that is also a basis for $\mathcal{H}$ ($k = \dim(\mathcal{H})$).

The <u>standard basis</u> of $\mathcal{H} = \mathbb{C}^n$ (n-dimensional column vectors)

For $1 \leq i \leq n$

$$e_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th position}$$

$\langle e_i, e_j \rangle = \delta_{ij}$

Generally,

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} v_i e_i$$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \sum_i u_i e_i$$

$$\langle u, v \rangle = \left\langle \sum_i u_i e_i, \sum_j v_j e_j \right\rangle$$

$$= \sum_j v_j \left\langle \sum_i u_i e_i, e_j \right\rangle$$

$$= \sum_j v_j \sum_i u_i^* \langle e_i, e_j \rangle$$

$$= \sum_{i,j} u_i^* v_j \langle e_i, e_j \rangle$$

$$= \sum_{i,j} u_i^* v_j \delta_{ij}$$

$$= \sum_i u_i^* v_i =: \langle u, v \rangle$$

$$= \underbrace{\begin{bmatrix} u_1^* & \cdots & u_n^* \end{bmatrix}}_{u^*} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

u^* (<u>adjoint</u>, covector of u)

$$= u^* v \quad (= \langle u, v \rangle)$$

<u>Def</u>: Let A be an $m \times n$ matrix over $\mathbb{C}$. The <u>adjoint</u> of A (<u>Hermitian conjugate</u> of A) is the $n \times m$ matrix obtained by transposing A (A^T) & taking the complex conjugate of each entry ("conjugate-transpose")

<u>Ex</u>: $A = \begin{bmatrix} 2i & 3-4i & 1+i \\ 7 & -2+i & 1-i \end{bmatrix}$

$$A^* = \begin{bmatrix} -2i & 7 \\ 3+4i & -2-i \\ 1-i & 1+i \end{bmatrix}$$

[Physicists tend to use $A^\dagger$ for the adjoint]

Properties:

$$(A+B)^* = A^* + B^*$$
$$(AB)^* = B^* A^*$$
$$a \in \mathbb{C} \quad (aA)^* = a^* A^*$$

Notation: If M is some matrix-valued expression, then $[M]_{ij}$ denotes the (i,j)th entry of M (ith row, jth column).

Notationally: $\forall i,j$

$$[A^*]_{ij} = [A]_{ji}^*$$

Basic Probability (discrete)

Ω is a finite or countably infinite set.

A probability distribution on Ω is a map

$Pr : \Omega \to \mathbb{R}$

such that, letting $p_x := Pr[x]$ for $x \in \Omega$,

$$\forall x \in \Omega, \ p_x \geq 0$$
$$\sum_{x \in \Omega} p_x = 1.$$

$\Omega = \{1, 2, 3, \ldots\}$

$Pr = (p_1, p_2, \ldots)$

A dist is deterministic if $p_x = 1$ for some $x \in \Omega$

[then $p_y = 0$ necessarily for all $y \neq x$]

Elements of Ω are called outcomes. Ω is a sample space. Ω with Pr is a probability space.

An event is any subset $E \subseteq \Omega$. Extend Pr to include events:

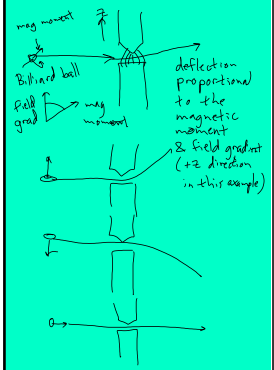
$$Pr[E] := \sum_{x \in E} Pr[x]$$

Ex:

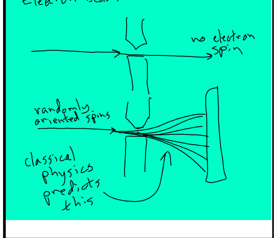
$Pr[\Omega] = 1$

$Pr[\emptyset] = 0$

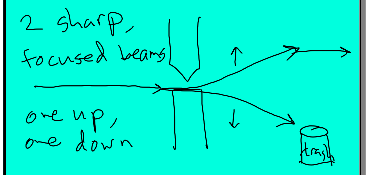
Stern-Gerlach experiment

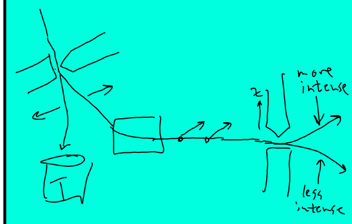


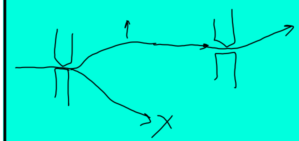
Electron beam



What actually happens:







Model electron spin states as vectors in $\mathbb{C}^2$:

e_1 — spin up

e_2 — spin down

Arbitrary vector in $\mathbb{C}^2$ can be written as

$\alpha e_1 + \beta e_2 =: u$

$(\alpha, \beta \in \mathbb{C})$

$\Pr[\uparrow] = |\alpha|^2 \; (= \alpha\alpha^*)$

$\Pr[\downarrow] = |\beta|^2 \; (= \beta\beta^*)$

α, β <u>probability</u> <u>amplitudes</u>

$(|\alpha|^2, |\beta|^2)$ is a prob dist on $\Omega = \{\uparrow, \downarrow\}$

$\therefore |\alpha|^2 + |\beta|^2 = 1$

$\quad\;\; \overset{\shortparallel}{\langle u, u \rangle}$ — u is a unit vector in $\mathbb{C}^2$.

spin states correspond to vectors in $\mathbb{R}^3$ on the unit sphere (<u>Bloch</u> <u>sphere</u>)