Complex numbers $\mathbb{C}$
- Re, Im, norm, arg
- mult, add
- exponential map
- algebraically closed

Linear algebra over $\mathbb{C}$
- inner product
- adjoints

NISQ era
(noisy intermediate-scale quantum)

A complex number is a number of the form

$z = x + iy$

where $x, y \in \mathbb{R}$ and i satisfies $i^2 = -1$ (imaginary unit).

x — $\mathrm{Re}(z)$ (real part)

y — $\mathrm{Im}(z)$ (imag part)

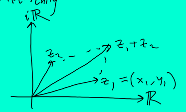
Associate z with the point (x, y) in $\mathbb{R}^2 \cong \mathbb{C}$ (complex plane)

Addition, multiplication

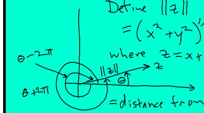
$$\overbrace{(x_1 + iy_1)}^{z_1} + \overbrace{(x_2 + iy_2)}^{z_2}$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

geometrically



Define $\|z\| = (x^2 + y^2)^{1/2}$ where $z = x + iy$.



$= $ distance from 0 to z

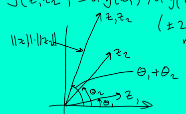
$\theta = \arg(z) = $ angle (counterclockwise) from pos $\mathbb{R}$-axis to z

Multiplication (geometically)

$$\|z_1 z_2\| = \|z_1\| \cdot \|z_2\|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$(\pm 2\pi n,\ n \in \mathbb{Z})$



Exponential map $\forall z \in \mathbb{C}$

$$e^z := \exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$

$$= \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n$$

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

$$e^0 = 1$$

$$e^{-z} = \frac{1}{e^z}$$

Complex conjugate:

Given $z = x + iy$,

$$z^* := \bar{z} = x - iy$$

$$z + z^* = 2x = 2\mathrm{Re}(z)$$

$$\mathrm{Re}(z) = \frac{z + z^*}{2}$$

$$\mathrm{Im}(z) = \frac{z - z^*}{2i}$$

$\frac{1}{i} = -i$

$$i \cdot (-i) = -(-1) = 1$$

$$zz^* = (x + iy)(x - iy)$$
$$= x^2 - (iy)^2$$
$$= x^2 + y^2 = \|z\|^2 \geq 0$$

$(= 0$ iff $x = y = 0$, i.e. $z = 0)$

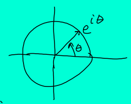
$$\|z\| = \sqrt{zz^*}$$

Euler's formula $\forall \theta \in \mathbb{R}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\|e^{i\theta}\| = (\cos^2\theta + \sin^2\theta)^{1/2} = 1$$

$e^{i\theta}$ is on the unit circle in $\mathbb{C}$



$e^{2\pi i} = 1$ exp has period $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

Algebraic closure.

Theorem (Fundamental Theorem of Algebra): $\mathbb{C}$ is algebraically closed.

Alg. closed means every univariate polynomial in $\mathbb{C}[X]$ of pos degree (non-constant) has a root in $\mathbb{C}$.

If $p(z)$ is nonconstant with coeffs in $\mathbb{C}$, then $\exists z_0$, $p(z_0) = 0$.

$\therefore p(z) = (z - z_0)q(z)$

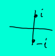
$\deg(q) = \deg(p) - 1$

etc.

$$\therefore p = \alpha \prod_{j=1}^{\deg(p)} (z - r_j)$$

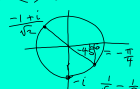
where $r_1, \dots, r_{\deg(p)}$ are the roots of p and $\alpha \in \mathbb{C}$, $\alpha \neq 0$.

Ex: $p(z) = z^2 + 1 = (z+i)(z-i)$



$p(z) = z^2 + i$

$p(z) = 0 \iff z^2 = -i$



$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \frac{1-i}{\sqrt{2}}$

Linear Algebra over $\mathbb{C}$

Def: An (n-dimensional) vector over $\mathbb{C}$ is an array of n elements of $\mathbb{C}$.

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \text{ — column vector}$$

$$[z_1 \cdots z_n] \text{ — row vector}$$

An $m \times n$ matrix is a rectangular array of cpx numbers with m rows and n columns

$$\begin{array}{cc} & n \text{ columns} \\ m \text{ rows} & \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & & \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix} \end{array}$$

A row vector is a matrix with $m = 1$

" column " " " with $n = 1$

Adding & multiplying matrices as normal

A 1×1 matrix $[a]$ identify with a (scalar)

Def: A $\mathbb{C}$-space $\mathcal{H}$ (finite dim Hilbert space) is a vector space of all n-dim column vectors with usual vector addition & scalar multiplication, together with an inner product $\langle \cdot, \cdot \rangle \to \mathbb{C}$ defined as

$$\langle u, v \rangle = \sum_{j=1}^{n} \underbrace{u_j^*}_{\text{complex conjugate}} v_j$$

where $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

Ex: $u = \begin{bmatrix} 3 \\ -4i \end{bmatrix}$, $v = \begin{bmatrix} -2+i \\ 6 \end{bmatrix}$

$$\langle u, v\rangle = 3(-2+i) + 4i \cdot 6$$
$$= -6 + 27i$$

Properties of $\langle \cdot, \cdot \rangle$:

$\langle \cdot, \cdot \rangle$ is linear in the 2nd argument: $\forall u, v, w \in \mathcal{H}$

$$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$$

and $\forall a \in \mathbb{C}$,

$$\langle u, av\rangle = a\langle u, v\rangle$$

verify:

$$\langle u, v+w\rangle = \sum_{j=1}^{n} u_j^*(v_j + w_j)$$
$$= \sum_{j=1}^{n} u_j^* v_j + \sum_{j=1}^{n} u_j^* w_j$$
$$= \langle u, v\rangle + \langle u, w\rangle$$

$$\langle u, av\rangle = \sum_j u_j^*(av_j) = a\sum_j u_j^* v_j$$
$$= a\langle u, v\rangle.$$

$\langle \cdot, \cdot \rangle$ is <u>conjugate symmetric</u>:

$$\langle u, v\rangle = \langle v, u\rangle^*$$

$$\langle v, u\rangle^* = \left(\sum_j v_j^* u_j\right)^*$$
$$= \sum_j (v_j^* u_j)^*$$
$$= \sum_j v_j^{**} u_j^*$$
$$= \sum_j u_j^* v_j$$
$$= \langle u, v\rangle$$

$(z_1+z_2)^* = z_1^* + z_2^*$

$(z_1 z_2)^* = z_1^* z_2^*$

$z^{**} = z$

<u>Exercise</u>: $\langle \cdot, \cdot \rangle$ is conjugate-linear in the 1st argument. That means

$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$

$$\langle au, v\rangle = a^*\langle u, v\rangle.$$

$\langle \cdot, \cdot \rangle$ is <u>positive-definite</u>

$\langle u, u\rangle \geq 0$ (& <u>real</u>)

with equality holding iff $u=0$

$$\langle u, u\rangle = \sum_{j=1}^{n} \underbrace{u_j^* u_j}_{\substack{\geq 0 \\ \& \ 0 \text{ iff } \\ u_j = 0}} > 0 \quad \text{if } u \neq 0.$$

$u \in \mathcal{H}$, define $\|u\| := \sqrt{\langle u, u\rangle}$

(analogous to $\|z\|$
$|z|$)