CSCE 785
12/7/2023

Bipartite pure-state entanglement (1)

Review

$\mathbb{C}$-spaces $\mathcal{H}$, $\mathcal{J}$. Pure state $\rho \in \mathcal{L}(\mathcal{H}\otimes\mathcal{J}) \cong \mathcal{L}(\mathcal{H})\otimes\mathcal{L}(\mathcal{J})$

$\rho = |\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle \in \mathcal{H}\otimes\mathcal{J}$

Def: ρ is separable if $\rho = \rho_{\mathcal{H}} \otimes \rho_{\mathcal{J}}$, where $\rho_{\mathcal{H}}$ and $\rho_{\mathcal{J}}$ are states over $\mathcal{H}$ and over $\mathcal{J}$ respectively. If ρ is pure, then so are $\rho_{\mathcal{H}}$ & $\rho_{\mathcal{J}}$. If ρ is not separable, we say ρ is entangled (between $\mathcal{H}$ & $\mathcal{J}$)

Note: $\operatorname{Tr}_{\mathcal{J}} \rho = \operatorname{Tr}_{\mathcal{J}}(\rho_{\mathcal{H}} \otimes \rho_{\mathcal{J}}) = (\operatorname{Tr}\rho_{\mathcal{J}})\rho_{\mathcal{H}} = \rho_{\mathcal{H}}$.
Likewise $\operatorname{Tr}_{\mathcal{H}} \rho = \rho_{\mathcal{J}}$.

Consider $\beta := |\Phi^+\rangle\langle\Phi^+|$ where $|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

So $\beta = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$

$\neq \rho \otimes \sigma$ for any 1-qubit states ρ, σ

β is entangled.

Theorem: $\mathcal{H}$, $\mathcal{J}$ $\mathbb{C}$-spaces. $|\psi\rangle \in \mathcal{H}\otimes\mathcal{J}$ unit vector.
There exist orthonormal sets $\{|\alpha_1\rangle, \ldots, |\alpha_k\rangle\} \subseteq \mathcal{H}$
and $\{|\gamma_1\rangle, \ldots, |\gamma_k\rangle\} \in \mathcal{J}$ (some unique k)

~~Such~~ and there exist unique real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ such that

$$\sum_{i=1}^{k} \lambda_i^2 = 1$$

and

$$|\psi\rangle = \sum_{i=1}^{k} \lambda_i \left(|\alpha_i\rangle \otimes |\gamma_i\rangle \right).$$

Def: This is known as the Schmidt decomposition of $|\psi\rangle$. ~~$\lambda_1, \ldots$~~ k is the Schmidt ~~number~~ rank of $|\psi\rangle$, and $\lambda_1, \ldots, \lambda_k$ are the Schmidt numbers or Schmidt coefficients of $|\psi\rangle$.

Note: $k \leq \min(\dim \mathcal{H}, \dim \mathcal{J})$.

Cor: $|\psi\rangle$ is separable iff $|\psi\rangle$ has Schmidt rank 1.

Proof of ~~(⇒)~~ $(\Leftarrow)$: $|\psi\rangle = |\alpha_1\rangle \otimes |\gamma_1\rangle$

So

$$\underbrace{|\psi\rangle\langle\psi|}_{\rho} = (|\alpha_1\rangle \otimes |\gamma_1\rangle)(\langle\alpha_1| \otimes \langle\gamma_1|)$$

$$= \underbrace{|\alpha_1\rangle\langle\alpha_1|}_{\rho_{\mathcal{H}}} \otimes \underbrace{|\gamma_1\rangle\langle\gamma_1|}_{\rho_{\mathcal{J}}}$$

$\therefore \rho$ is separable.

<u>Def</u>: Let $p := (p_1, \dots, p_k)$ be a probability distribution. The <u>(Shannon) entropy</u> of p is defined as

$$H(p) := -\sum_{j=1}^{k} p_j \log_2 p_j = \sum_{j=1}^{k} p_j \log_2 \frac{1}{p_j}$$

[<u>convention</u>: $0 \log_2 0 := 0 =: 0 \log_2 \infty$]

$H(p)$ is a measure of the uncertainty of p.

$H(p) \geq 0$ and $H(p) = 0$ iff $p_j = 1$ for some j.

$H(p) \leq \log_2 k$

$H(p) = \log_2 k$ iff p is <u>uniform</u>. ($p_j = \frac{1}{k}$ $\forall j$)

Given $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle \in \mathcal{H} \otimes \mathcal{J}$ unit vector.

Define the <u>amount of entanglement</u> of ρ to be $H(\lambda_1^2, \dots, \lambda_k^2)$ where $\lambda_1, \dots, \lambda_k$ are the Schmidt coeffs of $|\psi\rangle$.

<u>Cor</u>: ρ separable $\Longleftrightarrow \lambda_1^2 = 1$ $(k=1) \Longleftrightarrow$ ρ has 0 entanglement.

<u>Ex</u>: $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. $k=2$, $\lambda_1^2 = \lambda_2^2 = \frac{1}{2}$

$\rho = |\Phi^+\rangle\langle\Phi^+|$ has entanglement $\underbrace{H(\frac{1}{2}, \frac{1}{2})}_{\text{one bit}} = 1$

ρ has one e-bit of entanglement.

Def: Let ρ be a state. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of ρ

$\lambda_j \geq 0 \ \forall j$, $\sum \lambda_j = 1$

The ~~vonNe~~ von Neumann entropy of ρ is

$$S(\rho) = H(\lambda_1, \dots, \lambda_n).$$

says how "mixed" is ρ. 0 for pure states, positive for mixed states.

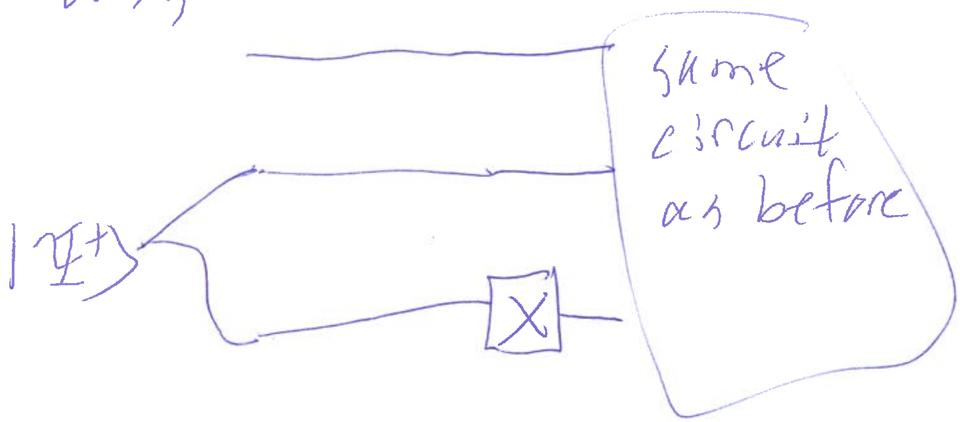
Thm (Cor to Schmidt decomp): $\rho \in \mathcal{L}(\mathcal{H} \otimes \mathcal{J})$ pure state. The amount of entanglement of ρ is equal to $S(Tr_{\mathcal{J}}\, \rho)$ and to $S(Tr_{\mathcal{H}}\, \rho)$.

~~Midterm~~ HW3 question

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

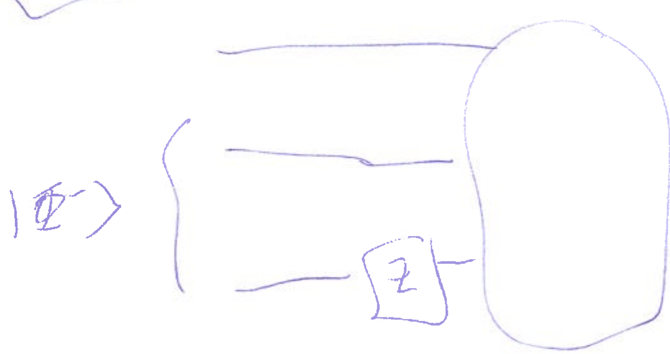
$$X_2|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

or X_1



$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Z_1 or Z_2 $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



Check wording of question