CSCE 785
12/5/2023

3 Topics: ①
- Stabilizer Codes (Ch 9)
- Fault-tolerant ("logical") gates
- Entanglement measures

Stabilizers — many n-qubit states can be described succinctly using stabilizers.

Fix $n \geq 1$. The <u>n-qubit</u> <u>Pauli</u> <u>group</u> Π^n is the set of all unitary operators in $\mathcal{L}((\mathbb{C}^2)^{\otimes n})$ of the form

$$\alpha(\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n)$$

where $\alpha \in \{1, -1, i, -i\}$ and each $\sigma_j \in \{I, X, Y, Z\}$.

This is a <u>group</u> (subgroup of unitary ops).

Finite with 4^{n+1} many elements.

<u>Fact</u>: $\forall g_1, g_2 \in \Pi^n$, either $g_1 g_2 = g_2 g_1$ or $g_1 g_2 = -g_2 g_1$.

<u>Def</u>: Let $G \leq \Pi^n$, let $|\psi\rangle$ be an n-qubit state. $|\psi\rangle$ is <u>stabilized</u> by G if $g|\psi\rangle = |\psi\rangle$ for all $g \in G$. Let $E_G := \{v : v \text{ is stabilized by } G\}$

Notice: E_G is a vector subspace of $(\mathbb{C}^2)^{\otimes n}$.

<u>Def</u>: Let G be a subgroup of Π^n. G is <u>stabilizing</u> if $-I \notin G$.

<u>Note</u>: if G not stabilizing, then $E_G = \{0\}$

<u>Prop</u>: If G is stabilizing, then $\dim(E_G) > 0$.

<u>Ex</u>: $n=3$. $|000\rangle \in E_G$ where $G = \langle Z_1, Z_2, Z_3 \rangle$
$\dim(E_G) = 1$

$= \langle Z \otimes I \otimes I, I \otimes Z \otimes I, I \otimes I \otimes Z \rangle$

subgroup of Π^n generated by these elements

b/c $Z|0\rangle = |0\rangle$

$|001\rangle$ stab by $\langle Z_1, Z_2, -Z_3 \rangle$

b/c $(-Z)|1\rangle = -(Z|1\rangle) = --|1\rangle = |1\rangle$

$|110\rangle$ stab by $\langle -Z_1, -Z_2, Z_3 \rangle$

etc.

$\dim(E_G) = 1$ in all cases above, so G determines the state up to an overall phase factor.

$\therefore$ Computational basis states can be ~~isolated~~ described by ~~their~~ stabilizing subgroups of Π^n.

$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\qquad$ $|-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$X|+\rangle = |+\rangle$ $\qquad$ $(-X)|-\rangle = |-\rangle$

So, e.g.

$|+++\rangle$ stab by $\langle X_1, X_2, X_3 \rangle$

$|++-\rangle$ " " $\langle X_1, X_2, -X_3 \rangle$

etc.

Def: $\{g_1, \ldots, g_k\} \subseteq \Pi^n$ is a minimal generating set for the group $\langle g_1, \ldots, g_k \rangle =: G$ if no smaller set of elements of Π^n generates G.

Thm: If $\{g_1, \ldots, g_k\}$ is a min. gen. set for a stabilizing subgroup $G \subseteq \Pi^n$, then $\dim(E_G) = 2^{n-k}$, and $|G| = 2^k$.

Stabilizer codes: The code space for a stabilizer code is of the form E_G for some stabilizing subgroup of Π^n.

The Shor Code is a stabilizer code.

3-qubit bit-flip code is a stabilizer code with code space spanned by $\{|000\rangle, |111\rangle\}$ and stab by $\cancel{\langle Z_1, Z_2, I \rangle}$

$$\begin{array}{c} ZZI \\ IZZ \end{array} = \langle Z_1Z_2, Z_2Z_3 \rangle = \langle Z\otimes Z\otimes I, I\otimes Z\otimes Z \rangle$$

3-qubit phase flip code space is spanned by $\{|+++\rangle, |---\rangle\}$, stabilized by

$$\begin{array}{c} XXI \\ IXX \end{array} = \langle X_1X_2, X_2X_3 \rangle = \langle X\otimes X\otimes I, I\otimes X\otimes X \rangle$$

Shor Code: $n = 9$

Stabilized by

$$\langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \rangle$$

9

	Z Z I	——	—— I
	I Z Z	I ——	—— I
	I I I	Z Z I	I I I
8	I I I	I Z Z	I I I
	I ——	—— I	Z Z I
	I ——	—— I	I Z Z
	X X X	X X X	I I I
	I I I	X X X	X X X

7-qubit CSS code

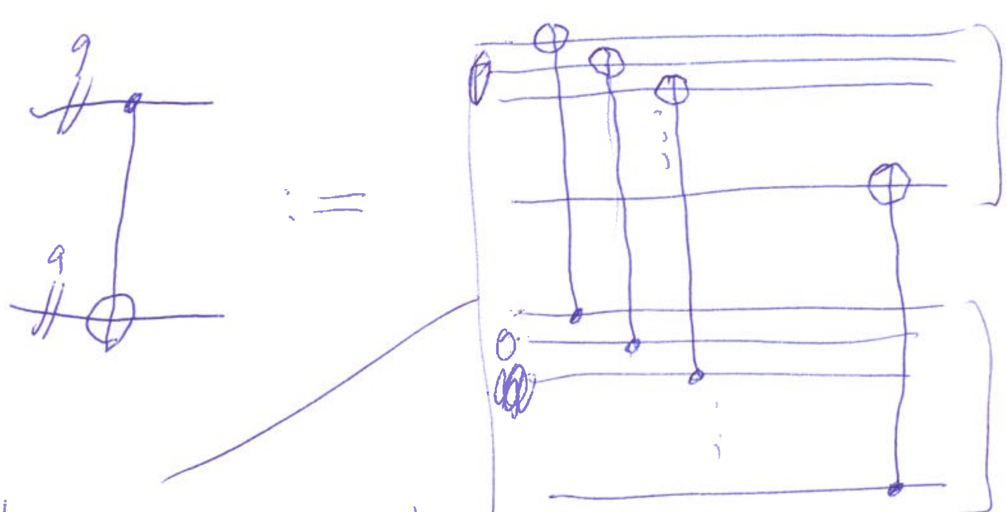
I I I Z Z Z Z
I Z Z I I Z Z
Z I Z I Z I Z

I I I X X X X
I X X I I X X
X I X I X I X

5-qubit code

X Z Z X I
I X Z Z X
X I X Z Z
Z X I X Z

Encoded gates for fault-tolerant computation



Shor-encoded registers

transverse circuit