CSCE 785
11/30/2023

Shor code (9-qubit code)

Error channel ~~(arb.)~~ 1-qubit error:

ρ 1-qubit state

$$\mathcal{D}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Z\rho Z + ZX\rho XZ) \quad [p < \tfrac{1}{2}]$$

$$= (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

(partially) "depolarizing channel"

L = "logical"

Bit-flip code: $|0\rangle \mapsto |0_L\rangle := |000\rangle$

$|1\rangle \mapsto |1_L\rangle := |111\rangle$

Phase-flip code: $|0\rangle \mapsto |+_L\rangle := \frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$

$|1\rangle \mapsto |-_L\rangle := \frac{1}{\sqrt{2}}(|0_L\rangle - |1_L\rangle)$

$$|+_L\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

$$|-_L\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

Shor code

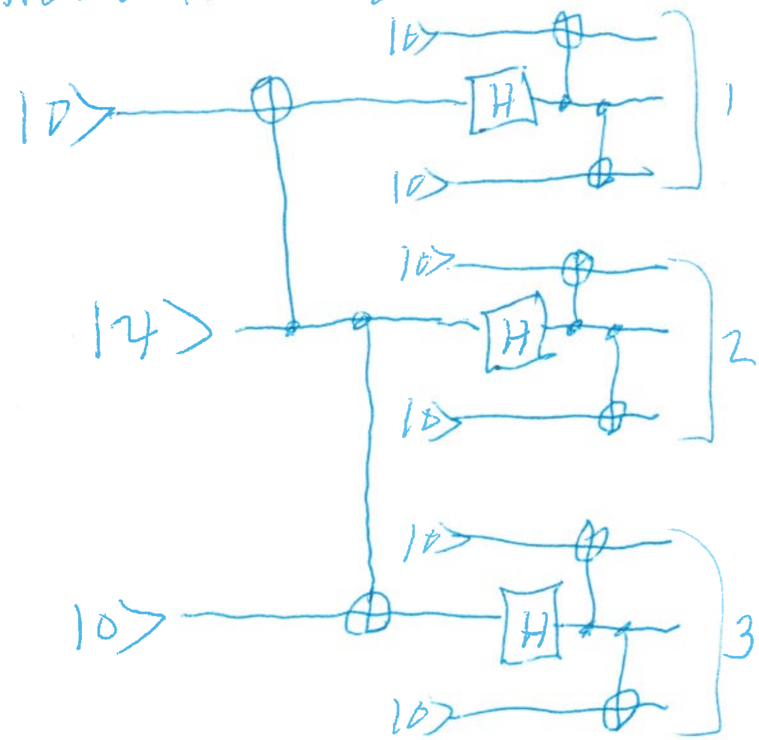
$$|0\rangle \mapsto |0_S\rangle := |+_L +_L +_L\rangle = |+_L\rangle^{\otimes 3}$$

$$= \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3}$$

$$|1\rangle \mapsto |1_S\rangle := |-_L -_L -_L\rangle = |-_L\rangle^{\otimes 3} = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}$$

(9 qubits

Encoding
Circuit for 1-qubit state $|\psi\rangle$



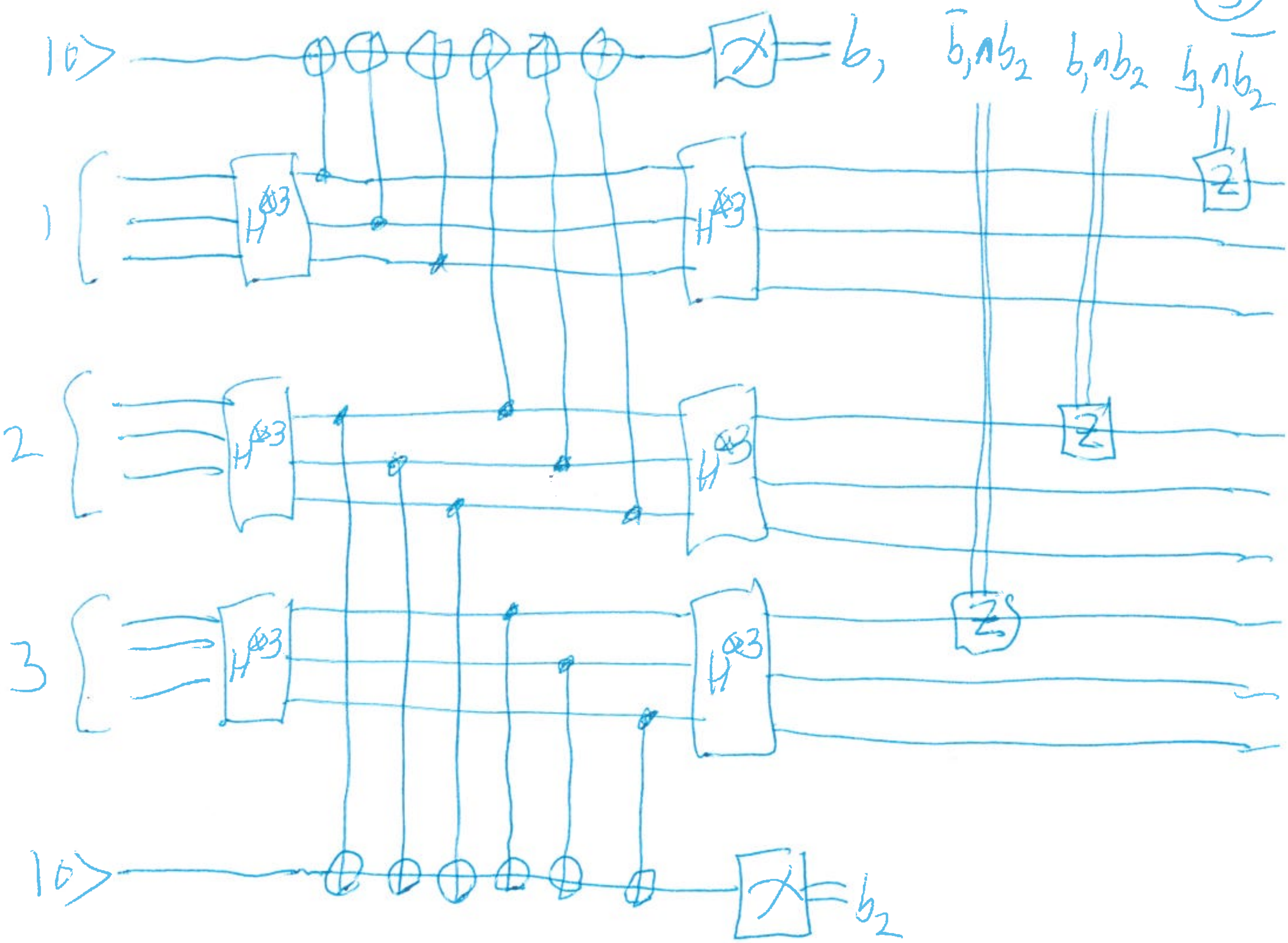
<u>Error correction</u>: 2 phases

1) Error-correct each 3-qubit block against bit flips

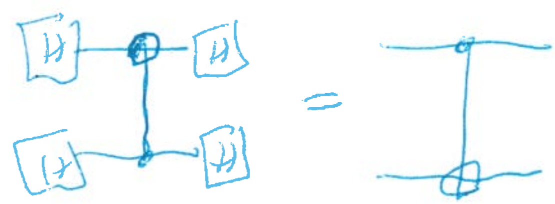
<u>Ex</u>: Block 1 $\quad \frac{1}{\sqrt{2}}\left(|\ldots\rangle \pm |\ldots\rangle\right) \xrightarrow[\text{correct}]{\text{error}} \frac{1}{\sqrt{2}}\left(|000\rangle \pm |111\rangle\right)$

equal bit values

2) Error correct phase flips — assuming that bit flips have been corrected in each block:



Can transform the circuit to reduce the number of H-gates



Error channel for the 9 qubits we assume to be

$$\mathcal{D}^{\otimes 9}(\rho) = (1-p)^9 \rho + (1-p)^8 \frac{p}{3} \sum_{j=1}^{9} \left(X_j \rho X_j + Y_j \rho Y_j + Z_j \rho Z_j \right) + O(p^2)$$

[ρ is 9-qubit state] [as $p \to 0$]

Prob[uncorrectable unencoded] $\approx O(p)$

Prob[" with Shor code] $\approx O(p^2)$

Discretization of Errors

Thm: Let $\mathcal{E}$ be a (not necessarily complete) error channel with Kraus operators

$$E_1, \ldots, E_k.$$

"not necessarily complete" means $\underbrace{\sum_{j=1}^{k} E_j^* E_j \leq I}$

$A \leq B$ means $B - A \geq 0$ positive semi-definite

Can complete an incomplete channel by ~~adding~~ including one more Kraus operator:

$$E_{k+1} := \sqrt{I - \sum_{j=1}^{k} E_j^* E_j}$$

Let $\mathcal{F}$ be an error channel (not nec complete) with Kraus operators $F_1, \ldots, F_\ell$ (some ℓ).

If each F_j is a linear combination of $E_1, \ldots, E_k$ and $\mathcal{E}$ is correctable, the $\mathcal{F}$ is correctable by the same procedure that corrects $\mathcal{E}$.

Corollary: The Shor code corrects arbitrary single-qubit errors.

Proof: Single-qubit error has Kraus ops

$$F \otimes I^{\otimes 8}, \text{ or } I \otimes F \otimes I^{\otimes 7} \text{ or } I^{\otimes 2} \otimes F \otimes I^{\otimes 6}, \ldots \text{ or } I^{\otimes 8} \otimes F$$

But X, Y, Z, I form a basis for 1-qubit operators. So, e.g.,

$$F \otimes I^{\otimes 8} = (\alpha_0 I + \alpha_1 X + \alpha_2 Y + \alpha_3 Z) \otimes I^{\otimes 8}$$

= lin comb of I, X_1, Y_1, Z_1

etc. $\blacksquare$