CSCE 785
11/21/2023

IBM final exercise: I'm thinking a teleportation circuit, but will entertain requests. ①

Quantum Channels (quantum operations)

Def: Let $\mathcal{H}$ and $\mathcal{J}$ be $\mathbb{C}$-spaces. A linear map "superoperator" $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{J})$ is a quantum channel if

1) Φ is trace-preserving (i.e., $\operatorname{tr}(\Phi(A)) = \operatorname{tr} A$ for all $A \in \mathcal{L}(\mathcal{H})$.)

2) Φ is completely positive.

Idea: A quantum channel maps states to states (pos op. with $\operatorname{tr}=1$ $\longmapsto$ pos. op with $\operatorname{tr}=1$)

Def: $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{J})$ is positive if $\forall A \in \mathcal{L}(\mathcal{H})$, $A \geq 0 \implies \Phi(A) \geq 0$.

Def: Φ is completely positive if, $\forall \mathbb{C}$-space $\mathcal{K}$ $\Phi \otimes \mathbb{1}_{\mathcal{K}}$ is positive. [in fact, $\Phi \otimes \mathbb{1}_{\mathcal{K}}$ is completely positive]

[$\mathbb{1}_{\mathcal{K}} : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{K})$ is the identity on $\mathcal{L}(\mathcal{K})$, so $\Phi \otimes \mathbb{1}_{\mathcal{K}} : \mathcal{L}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{L}(\mathcal{J} \otimes \mathcal{K})$ is such that

$$(\Phi \otimes \mathbb{1}_{\mathcal{K}})(A \otimes B) = \Phi(A) \otimes B, \quad A \in \mathcal{L}(\mathcal{H}) \ \& \ B \in \mathcal{L}(\mathcal{K})$$

& extend uniquely by linearity]

Idea: Attaching the trivial channel (the identity) to Φ still gives a positive map.

Examples: (1) Let $U \in \mathcal{L}(\mathcal{H})$ be unitary.

$$\Phi_U : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$$

$$A \mapsto UAU^* \qquad \text{(unitary conjugation)}$$

"unitary channel" models evolution of an isolated system.

(2) ~~Φ~~ Fix a state $\rho \in \mathcal{L}(\mathcal{H})$.

$$\Phi : \mathbb{C} \to \mathcal{L}(\mathcal{H})$$

$$\cong \mathcal{L}(\mathbb{C})$$

$$z \in \mathbb{C} \mapsto z\rho$$

"state preparation channel"

e.g., $1 \mapsto \rho$

(3) Projective measurement: Let $\{P_1, \ldots, P_k\} \subseteq \mathcal{L}(\mathcal{H})$ ~~pos~~ csop. Fix a k-dimensional $\mathbb{C}$-space Θ ("outcomes")

$$\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H} \otimes \Theta)$$

$$A \mapsto \sum_{j=1}^{k} P_j A P_j \otimes E_{jj}$$

$E_{jj} \in \mathcal{L}(\Theta)$ diagonal, $E_{jj} = |j\rangle\langle j|$

$$= \sum_{j=1}^{k} \underbrace{\Pr\{j\}}_{\text{prob. of outcome } j} \underbrace{\left(\frac{P_j A P_j}{\Pr\{j\}} \right)}_{\substack{\text{post-measurement} \\ \text{state given outcome } j}} \otimes E_{jj}$$

<u>Partial Trace channel.</u> $\mathcal{H}, \mathcal{J}$ $\mathbb{C}$-spaces.

The unique linear map $\mathrm{Tr}_{\mathcal{J}} : \mathcal{L}(\mathcal{H} \otimes \mathcal{J}) \to \mathcal{L}(\mathcal{H})$

satisfying $\mathrm{Tr}_{\mathcal{J}}(A \otimes B) = (\mathrm{tr}\, B) A$ for all $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{J})$

is called ~~the~~ a <u>partial trace</u>. [this is a channel]

Applying $\mathrm{Tr}_{\mathcal{J}}$ is sometimes called "tracing out $\mathcal{J}$."

<u>Example</u>: $\rho \in \mathcal{L}(\mathcal{H})$ and $\sigma \in \mathcal{L}(\mathcal{J})$ are states,

then $\rho \otimes \sigma$ is a state in $\mathcal{L}(\mathcal{H} \otimes \mathcal{J})$. And

$$\mathrm{Tr}_{\mathcal{J}}(\rho \otimes \sigma) = (\mathrm{tr}\, \sigma)\rho = \rho$$

$\mathrm{Tr}_{\mathcal{J}}$ means "ignore system $\mathcal{J}$"

Likewise: $\mathrm{Tr}_{\mathcal{H}} : \mathcal{L}(\mathcal{H} \otimes \mathcal{J}) \to \mathcal{L}(\mathcal{J})$

$$A \otimes B \mapsto (\mathrm{tr}\, A) B$$

"tracing out $\mathcal{H}$"

Representations of channels:

1) Operator sum
2) Stinespring
3) Choi

For every channel $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{J})$ there exist a finite set $K_1, \ldots, K_k \in \mathcal{L}(\mathcal{H}, \mathcal{J})$ such that, ~~$\forall A \in \mathcal{L}(\mathcal{H})$~~

$$\sum_{j=1}^{k} K_j^* K_j = \mathbb{1}_{\mathcal{H}} \qquad (*)$$

and $\forall A \in \mathcal{L}(\mathcal{H})$,

$$\Phi(A) = \sum_{j=1}^{k} K_j A K_j^* . \qquad (**)$$

Conversely any Φ satisfying $(*)$ and $(**)$ for some operators $K_1, \ldots, K_k$ is a channel.

$K_1, \ldots, K_k$ are called <u>Kraus operators</u>.

<u>Ex</u>: $K_1 = U$ (unitary) $\in \mathcal{L}(\mathcal{H})$

$$\Phi(A) = \sum_{j=1}^{1} K_j A K_j^* = UAU^*$$

$$\left[\text{and } \sum_{j=1}^{1} K_j^* K_j = U^* U = \mathbb{1}_{\mathcal{H}} \right]$$

Model noise by a quantum channel (noise channel)

$\mathcal{H} = \mathbb{C}^2 = J$, let $p \in [0,1]$. Channel

$$\mathcal{L}(\mathbb{C}^2) \longrightarrow \mathcal{L}(\mathbb{C}^2)$$
$$A \longmapsto (1-p)A + pXAX \qquad \left[X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]$$

"bit flip channel" models a bit flip with prob p

Kraus operators: $\{\sqrt{1-p}\, I, \sqrt{p}\, X\}$

"Phase flip channel" $\mathcal{H} = \mathbb{C}^2 = J$

$$A \in \mathcal{L}(\mathbb{C}^2) \longmapsto (1-p)A + pZAZ \qquad \left[Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right]$$

phase flip with probability p.

Combined bit/phase flip:

$$A \longmapsto (1-p)A + \frac{p}{3}XAX + \frac{p}{3}ZAZ + \frac{p}{3}ZXAXZ$$

$$= (1-p)A + \frac{p}{3}XAX + \frac{p}{3}ZAZ + \underbrace{\frac{p}{3}XZAZX}_{\frac{p}{3}YAY}$$

suffices to model an arbitrary 1-qubit error!