CSCE 785
11/16/2023

Basic Quantum Information

Norms of operators

Recall: $A \in \mathcal{L}(\mathcal{H})$, define $|A| = \sqrt{A^*A}$

Eigenvalues of $|A|$ are the singular values of A.

all ≥ 0. Let $s_1, \ldots, s_n$ be the singular values of A

$s_j \geq 0$ $\forall j$. Let $p \geq 1$ $(p \in \mathbb{R})$

Def:

$$\|A\|_p := \left(\sum_{j=1}^{n} s_j^p\right)^{1/p} = \left(\mathrm{Tr}|A|^p\right)^{1/p}$$

the Schatten p-norm of A

" ℓ_p-norm of A.

Ex: $p=2$: $\|A\|_2 = \left(\mathrm{Tr}(|A|^2)\right)^{1/2} = \sqrt{\mathrm{Tr}(A^*A)} = \sqrt{\langle A, A\rangle}$

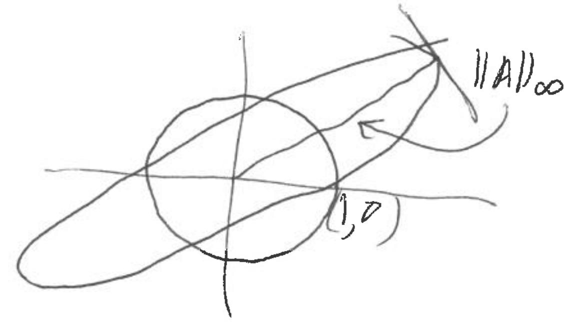
(Euclidean norm, Frobenius norm, ℓ_2-norm, Hilbert-Schmidt norm)

$p=1$: $\|A\|_1 = \mathrm{Tr}|A| = \sum_j s_j$ Trace norm

$p=\infty$: $\|A\|_\infty = \|A\| = \lim_{p\to\infty} \cancel{\|A_p\|} \|A\|_p$ $\left\{ \begin{array}{l} \ell_\infty\text{-norm}, \\ \text{operator norm} \\ = \max(s_1, \ldots, s_n) \end{array} \right.$

Can Show: $p \leq q \Rightarrow \|A\|_p \geq \|A\|_q$

$$\underbrace{\|A\|}_{\|A\|_\infty} = \sup_{\|v\|=1} \underbrace{\|Av\|}_{\substack{\text{norm}\\ \text{in } \mathcal{H}}}.$$



"General" measurements: Most general measurement on a quantum system (don't care about post-meas state)

Positive Operator-Valued Measure (POVM)

Collection of positive ($\Rightarrow$ Hermitian) operators $\{M_j \in \mathcal{L}(\mathcal{H})\}_{j\in S}$

S is a finite (or countably infinite) set of possible outcomes, where $M_j \geq 0 \;\; \forall j$

and $\sum_{j\in S}' M_j = \mathbb{1}$

A projective measurement is an example of a POVM.

Given a quantum state $\rho \in \mathcal{L}(\mathcal{H})$ [say, $\rho = |\psi\rangle\langle\psi|$] & POVM $\{M_j\}$ as above, measuring ρ yields outcome j with prob

$$\Pr[j] = \langle M_j, \rho\rangle = \mathrm{Tr}(M_j \rho) \geq 0.$$

Probs sum to 1:

$$\sum_{j\in S}' \Pr[j] = \sum_{j\in S}' \langle M_j, \rho\rangle = \left\langle \sum_j' M_j , \rho \right\rangle = \langle \mathbb{1}, \rho\rangle = \mathrm{Tr}\,\rho = 1.$$

Only 2 properties of ρ needed: $\rho \geq 0$ and $\operatorname{Tr} \rho = 1$

Generalize the concept of a quantum state:

We will say $\rho \in \mathcal{L}(\mathcal{H})$ is a <u>quantum state</u> if $\rho \geq 0$ & $\operatorname{Tr} \rho = 1$.

<u>Ex</u>: $\rho = |\psi\rangle\langle\psi|$ pure state. ($|\psi\rangle$ is a unit vector)

$$\rho = (|\psi\rangle)^* |\psi\rangle \geq 0$$

$$\operatorname{Tr} \rho = \operatorname{Tr}(|\psi\rangle\langle\psi|) = \operatorname{Tr}\langle\psi|\psi\rangle = \langle\psi|\psi\rangle = 1.$$

<u>Generally</u>: Any convex combination of states is a state.

$A_1, \dots, A_k$ a convex combo. of $A_1, \dots, A_k$ is a linear combo $p_1 A_1 + \cdots + p_k A_k$ where $(p_1, \dots, p_k)$ are a prob distribution ($p_j \geq 0$, $\sum_j p_j = 1$)

$$\rho := \sum_j \lambda_j \rho_j \qquad (\rho_j \text{ states})$$

ρ is a state

$$\rho = \sum_j \lambda_j \rho_j \geq 0 \quad b/c \quad \lambda_j \geq 0 \quad \forall j$$

$$\operatorname{Tr} \rho = \sum_j \operatorname{Tr}(\lambda_j \rho_j) = \sum_j \lambda_j \operatorname{Tr} \rho_j = \sum_j \lambda_j = 1.$$

Def: A mixed state is any state that is not of the form $|\psi\rangle\langle\psi|$ for unit $|\psi\rangle$, i.e., not a pure state.

Mixed states = nontrivial convex combo's of pure states.

Prop: Let $\rho \in \mathcal{L}(\mathcal{H})$ ($\dim \mathcal{H} = n$) be a state. There exist unit vectors $|\psi_1\rangle, \ldots, |\psi_n\rangle$ & prob dist $(p_1, \ldots, p_n)$ such that

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| + \cdots + p_n|\psi_n\rangle\langle\psi_n|.$$

Moreover, $\langle\psi_i|\psi_j\rangle = 0$ for all $i \neq j$.

Pf: Take $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ to be an eigenbasis for ρ (eigenval of $|\psi_j\rangle$ is p_j). Then $\sum_j p_j|\psi_j\rangle\langle\psi_j|$ is ~~the~~ a spectral decomp of ρ.

Scenario: Alice prepares one of the states in $\{\rho_1, \ldots, \rho_k\}$ (arbitrary set of states) randomly, choosing ρ_j with some prob. λ_j, sends the state $\sigma := \rho_j$ to Bob.

What can Bob find out?

Most generally, Bob can measure σ with some POVM $\{M_\ell\}_{\ell \in S}$. Then $\forall \ell \in S$,

$$\Pr[\ell] = \sum_{j=1}^{k} \underbrace{\Pr[\ell \mid \sigma = \rho_j]}_{\langle M_\ell, \rho_j \rangle} \cdot \underbrace{\Pr[\sigma = \rho_j]}_{\lambda_j}$$

$$= \sum_{j=1}^{k} \lambda_j \langle M_\ell, \rho_j \rangle = \left\langle M_\ell, \sum_{j=1}^{k} \lambda_j \rho_j \right\rangle$$

$$= \langle M_\ell, \rho \rangle \quad \text{where } \rho := \sum_{j=1}^{k} \lambda_j \rho_j$$

$\therefore$ Bob's statistics only depend on the state ρ.

Another scenario:

Alice's Plan A: prepare a qubit either in the

state $\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle$
or $\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle$] prob $\frac{1}{2}$ for each

sends resulting state to Bob.

Alice's Plan B: prepare a qubit either in state

$|0\rangle$ with prob $\frac{16}{25}$

or $|1\rangle$ with prob $\frac{9}{25}$

sends result to Bob.

Plan A: The resulting mixed state is

$$= \frac{1}{2}\left(\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle\right)\left(\frac{4}{5}\langle 0| + \frac{3}{5}\langle 1|\right) + \frac{1}{2}\left(\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle\right)\left(\frac{4}{5}\langle 0| - \frac{3}{5}\langle 1|\right)$$

$$= \frac{16}{25}|0\rangle\langle 0| + \frac{9}{25}|1\rangle\langle 1|$$

equal!

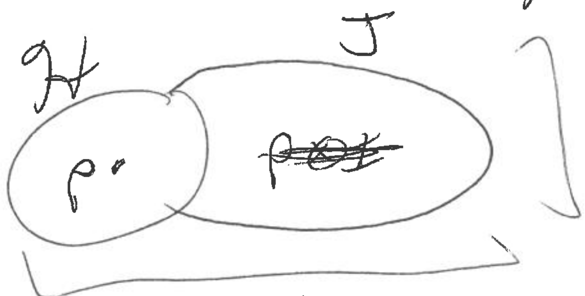
Plan B state: $\frac{16}{25}|0\rangle\langle 0| + \frac{9}{25}|1\rangle\langle 1|$

Any POVM can be accomplished by

1) ~~emb~~ attaching an auxillary physical system

1.5) evolving combined system unitarily

2) doing a projective measurement in the combined system.



$$\rho\otimes I$$
$$\downarrow$$
$$U(\rho\otimes I)U^*$$
$$\downarrow \text{ proj measurement}$$