CSCE 785
11/7/23

~~CSCE~~ Consequences of the Spectral Theorem (1)

- Unique spectral decomp
- Functions of an operator:
- Square root
- Absolute value
- Positive Operators

Spectral Theorem says: "Any property or function you can define on diagonal matrices that are invariant under unitary conjugation also hold for all normal operators."

__Theorem__ (spectral decomposition): If $A \in \mathcal{L}(\mathcal{H})$ is normal then there exists a unique set $\{(\lambda_1, P_1), (\lambda_2, P), \ldots, (\lambda_k, P_k)\}$ such that $\lambda_i \in \mathbb{C}$, P_i all projectors

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k,$$

and $\{\lambda_1, \ldots, \lambda_k\}$ are the distinct eigenvalues of A and each P_i projects orthogonally onto the corresponding eigenspace. Moreover $\{P_1, \ldots, P_k\}$ is a csop.

$[\dim(\mathcal{E}_{\lambda_i}(A)) = \operatorname{tr} P_i = \operatorname{rank} P_i]$

→ Example: "P is a projector" is unitary conjugation-invariant, that is, P is a projector iff UPU^* is a projector for any unitary U.

A diagonal matrix is a projector iff ______ ?

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} = \begin{bmatrix} a_1^2 & & 0 \\ & \ddots & \\ 0 & & a_n^2 \end{bmatrix} = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} = P$$

So $a_j^2 = a_j$ for all j, $\therefore a_j = 0$ or 1.

~~tr P~~ $\operatorname{tr} P = \#$ of 1's on the diagonal $= \operatorname{rank} P$.

Functions of an operator.

Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ some function

Let $A \in \mathcal{L}(\mathcal{H})$ be normal such that

$\operatorname{spec}(A) \subseteq \Omega$ [f well-defined on A's eigenvalues].

Then we define

$$f(A) := f(\lambda_1)P_1 + f(\lambda_2)P_2 + \cdots + f(\lambda_k)P_k$$

where $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$ is the unique spec. decomp on the last page.

Fact: For any unitary U, $f(UAU^*) = U f(A) U^*$

Pf: $\cancel{f(UAU^*)} = U f(A) U^* = \underbrace{f(\lambda_1)UP_1U^* + \cdots + f(\lambda_k)UP_kU^*}_{\cancel{\text{unique spect}}}$

and check that $\lambda_1 UP_1U^* + \cdots + \lambda_k UP_kU^*$ is the unique spec. decomp of UAU^* //

When is a diagonal matrix unitary?

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}^* = \begin{bmatrix} a_1 a_1^* & & 0 \\ & \ddots & \\ 0 & & a_n a_n^* \end{bmatrix} = I$$

if unitary

$a_j a_j^* = |a_j|^2 = 1$ $\therefore$ All eigenvals of any unitary matrix have unit norm in $\mathbb{C}$ (of the form $e^{i\theta}$, $\theta \in \mathbb{R}$)

When is a diagonal matrix Hermitian?

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} = \begin{bmatrix} a_1^* & & 0 \\ & \ddots & \\ 0 & & a_n^* \end{bmatrix}$$

$\therefore a_j = a_j^*$, equiv: $a_j \in \mathbb{R}$

Def: Let $A \in \mathcal{L}(\mathcal{H})$. A is positive semidefinite (or just positive), written $A \geq 0$, if $A = B^* B$ for some B.

Fact: $A \geq 0 \Rightarrow A$ is normal

Fact: $A \geq 0 \Leftrightarrow \langle v, Av \rangle \geq 0$ for any $v \in \mathcal{H}$

Def: A is strictly positive (or positive definite), written $A > 0$ if $A \geq 0$ and A is invertible.

A normal operator is ...	... iff all its eigenvalues are ...
unitary	on the unit circle in $\mathbb{C}$
Hermitian	real
positive	≥ 0
strictly positive	> 0
a projector	0 or 1

Functions of an operator (again)

If $A \geq 0$, then define $\sqrt{A}$ as

$$\sqrt{A} = \sqrt{\lambda_1}\, P_1 + \cdots + \sqrt{\lambda_k}\, P_k$$ by the unique spectral decomp.

Fact: For $A \geq 0$, $\sqrt{A}$ is the unique ~~B~~ $B \geq 0$ such that $B^2 = A$.

Prop: ~~A~~ If $A \geq 0$, then $\langle A, B \rangle \geq 0$ for all $B \geq 0$ with equality holding iff $AB = 0$.

Proof: $\langle A, B \rangle = \mathrm{Tr}(A^* B) = \mathrm{Tr}(AB) = \mathrm{Tr}(\sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B})$

$= \mathrm{Tr}(\sqrt{B}\sqrt{A}\sqrt{A}\sqrt{B}) = \mathrm{Tr}((\sqrt{A}\sqrt{B})^* \sqrt{A}\sqrt{B}) = \langle C, C \rangle \geq 0$

[where $C := \sqrt{A}\sqrt{B}$]

and $\langle C, C\rangle = 0 \Rightarrow C = 0 = \sqrt{A}\sqrt{B}$

$$\therefore AB = \sqrt{A}\underbrace{\sqrt{A}\sqrt{B}}_{0}\sqrt{B} = 0$$

[Can't just square both sides of $0 = \sqrt{A}\sqrt{B}$ because $\sqrt{A}$ and $\sqrt{B}$ don't necessarily commute.]

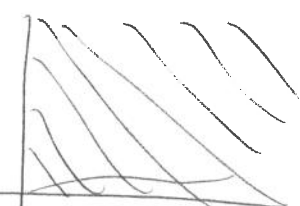
Finally, if $AB = 0$, then $\langle A, B\rangle = Tr(AB) = Tr\, 0 = 0$.

Def: A cone in any $\mathbb{C}$-space is a set C that is closed under multiplication by positive scalars:

$$(A \in C \text{ and } a > 0 \text{ then } aA \in C)$$

Clearly, pos operators from a cone.

Ex in $\mathbb{R}^2$



pos quadrant

$x \in \mathbb{R}^2$

x in pos quad iff $x \cdot y \geq 0$ for any y in the quadrant

Positive operators have similar property

Def: Let $A \in \mathcal{L}(\mathcal{H})$ be arbitrary. Define

$$|A| = \sqrt{A^* A}$$

Fact: If $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$ unique spec decomp, then

$$|A| = |\lambda_1| P_1 + \cdots + |\lambda_k| P_k$$