CSCE 785 | SRI | Revised | 1

11/2/2023

Normal operators & the spectral theorem

Recall: From Shor's algo: Find least k, r s.t. $\left|\frac{y}{2^{2n}} - \frac{k}{r}\right| \le \underbrace{2^{-n-1}}_{\varepsilon}$

Equiv: $\frac{k}{r} \in \left[\frac{y}{2^{2n}} - \varepsilon, \frac{y}{2^{2n}} + \varepsilon\right]$

$\frac{k}{r}$ is the SRI of $\left(\frac{y}{2^{2n}} - \varepsilon, \frac{y}{2^{2n}} + \varepsilon\right)$ (simplest rational interpolant)

SRI(a,b) $\quad a,b \in \mathbb{R},\ 0 < a < b$

// return $\frac{n}{d}$ $\quad n,d \in \mathbb{Z}$ s.t. d is least pos int s.t. $\exists n,\ \frac{n}{d} \in [a,b]$ & then pick n to be the least such n.

if $a \le 1 \le b$ then return $n := 1, d := 1$ (return $\frac{1}{1}$)

elsif $a > 1$ then

  let $q := \lceil a - 1 \rceil$ // $q \in \mathbb{Z}$ is largest such that $q < a$

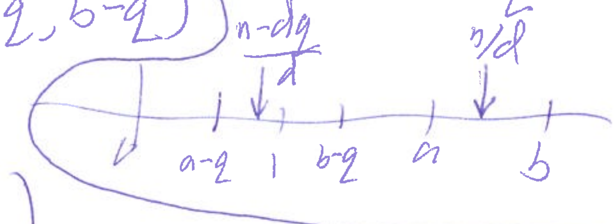
  return $q + \text{SRI}(a-q, b-q)$

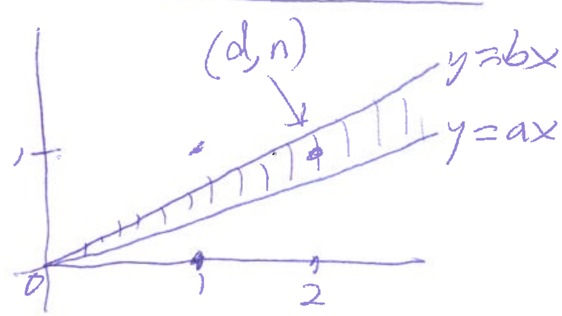
else // $0<a<b<1$

  let $\frac{n}{d} = \text{SRI}\left(\frac{1}{b}, \frac{1}{a}\right)$

  return $\frac{d}{n}$

end





Normal operators $\dim(\mathcal{H}) = n$

Def: An operator $A \in \mathcal{L}(\mathcal{H})$ (or $n \times n$ matrix) is normal if $AA^* = A^*A$.

Ex:
- Hermitian operators & unitary operators are normal.
- Diagonal matrices are normal.
- "is normal" is invariant under unitary conjugation:

If A is normal then UAU^* is normal for any unitary U: [Proof: Assume A is normal

$$(UAU^*)^*(UAU^*) = UA^*U^*UAU^*$$
$$= UA^*AU^* = UAA^*U^* = UAU^*UA^*U^*$$
$$= (UAU^*)(UAU^*)^* \quad //$$
]

$\therefore UAU^*$ is normal.

Def: $\chi_A(\lambda) = \det(A - \lambda I)$ degree n polynomial in λ.

Roots of χ_A are the eigenvalues of A

{$v \neq 0$ is an eigenvector of A with eigenval λ if $Av = \lambda v$}

The (multi)set of eigenvalues of A is the spectrum of A ($\mathrm{spec}(A)$).

Ex: If A is triangular, then $\mathrm{spec}(A)$ is the set of diagonal entries:

$$\det(A - \lambda I) = \prod_{j=1}^{n} ([A]_{jj} - \lambda)$$

Also, $\chi_A = \chi_{UAU^*}$ for unitary U.

[In fact $\chi_{AB} = \chi_{BA}$.] ③

<u>Def</u>: Let $A \in \mathcal{L}(\mathcal{H})$ and let λ be an eigenval of A. Def $\mathcal{E}_\lambda(A) = \{v : Av = \lambda v\}$ (subspace of $\mathcal{H}$) is the <u>eigenspace</u> of A corresp. to λ.

<u>Theorem</u> (Spectral Theorem for normal operators):
If $A \in \mathcal{L}(\mathcal{H})$ is normal, then there exists a diagonal matrix D and a unitary U such that $A = UDU^*$ (equivalently $U^*AU = D$)
{<u>Equivalently</u>: $\exists$ an ortho. basis of eigenvectors of A.}
~~Further~~ [It follows that the entries of D's diagonal are the eigenvals of A]

<u>Lemma</u>: Let M be a normal $n \times n$ matrix over $\mathbb{C}$.
Let $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for some square $\underset{r \times r}{A}$ & $\underset{s \times s}{D}$, $r+s=n$

If $B = 0$, then $C = 0$ and A & D are both normal.

<u>Proof</u>: Assume $B=0$

$$MM^* = \left[\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} A^* & C^* \\ \hline 0 & D^* \end{array}\right] = \left[\begin{array}{c|c} - & - \\ \hline - & CC^* + DD^* \end{array}\right]$$

$$\parallel$$

$$M^*M = \left[\begin{array}{c|c} A^* & C^* \\ \hline 0 & D^* \end{array}\right]\left[\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} - & - \\ \hline - & D^*D \end{array}\right]$$

$$\therefore\ CC^* + DD^* = D^*D$$

$$\therefore\ \underbrace{tr(CC^* + DD^*)}_{\parallel} = tr(D^*D)$$

$$tr(CC^*) + \cancel{tr(DD^*)} = tr(D^*D) = \cancel{tr(DD^*)}$$

$$\therefore\ tr(CC^*) = 0 = tr(C^*C) = \langle C, C\rangle \quad \text{pos def}$$

$$\therefore\ C = 0.$$

$$\therefore\ M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right]$$

$$\left[\begin{array}{c|c} AA^* & 0 \\ \hline 0 & DD^* \end{array}\right] = MM^* = M^*M = \left[\begin{array}{c|c} A^*A & 0 \\ \hline 0 & D^*D \end{array}\right]$$

$$\therefore\ AA^* = A^*A \quad \text{and} \quad DD^* = D^*D$$

$\therefore$ A, D normal $\boxed{///}$

<u>Proof of the spectral thm</u>: Induction on n.

$n = 1$: nothing to prove.

$n > 1$: Let λ be an eigenvalue of A, (then λ^* is an eigenval of A^*) and let v be an eigenvector of A^* with eigenval λ^* such that $\|v\| = 1$. Then $\exists$ unitary U such that

$$\text{that } \underbrace{U^*AU}_{A'} = \left[\begin{array}{c|ccc} \lambda & 0 & \cdots & 0 \\ \hline & & & \\ & & \hat{A} & \\ & & & \end{array}\right]$$

(see page 5(*))

~~Choose U so that~~

By inductive hyp, $\exists$ unitary $\hat{U}$ $((n-1)\times(n-1))$ such that $\hat{U}^*\hat{A}\hat{U}$ is diagonal $\hat{D}$

then let $V = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{U} \end{array}\right]$. V is unitary

and

$$V^*\left[\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \hat{A} \end{array}\right]V = \left[\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \hat{U}^*\hat{A}\hat{U} \end{array}\right] = \left[\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \hat{D} \end{array}\right]$$

$$\parallel$$

$$V^*U^*AUV = (UV)^*A\underbrace{(UV)}_{\text{unitary}}$$

(*) To get ~~U^*AU~~ $U^*AU = \left[\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline & \hat{A} \end{array}\right]$ this is equivalent to $\boxed{e_1^*U^*AU = \lambda e_1^*}$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

(so $e_1^* = [1\ 0 \cdots 0]$)

Choose U so that $Ue_1 = v$. Then

$$e_1^*U^*AU = (Ue_1)^*AU = v^*AU = (A^*v)^*U = (\lambda^*v)^*U$$

$$= \lambda v^*U = \lambda(Ue_1)^*U = \lambda e_1^*U^*U = \lambda e_1^*.$$