CSCE 785
10/10/23

1-qubit unitaries, Euler Angles
Deutsch-Josza circuit
Simon's Problem

Def: $\forall \theta \in \mathbb{R}$,

1-qubit unitaries:

$$R_x(\theta) := e^{-iX\theta/2} = \cos\left(\tfrac{\theta}{2}\right) I - \left(i \sin \tfrac{\theta}{2}\right) X$$

$$R_y(\theta) := e^{-iY\theta/2} = \left(\cos \tfrac{\theta}{2}\right) I - \left(i \sin \tfrac{\theta}{2}\right) Y$$

$$R_z(\theta) := e^{-iZ\theta/2} = \left(\cos \tfrac{\theta}{2}\right) I - \left(i \sin \tfrac{\theta}{2}\right) Z$$

$R_x(\theta), R_y(\theta), R_z(\theta)$ corresponds to a (counterclockwise) rotation about the x-, y-, or z-axis, respectively.

Note: $R_x(\pi) = -iX \propto X$, $R_y(\pi) = -iY \propto Y$, $R_z(\pi) = -iZ \propto Z$

Also: $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \propto R_z\left(\tfrac{\pi}{2}\right)$, $T = \begin{bmatrix} 1 & 0 \\ 0 & \underbrace{\tfrac{1+i}{\sqrt{2}}}_{= e^{i\pi/4}} \end{bmatrix} \propto R_z\left(\tfrac{\pi}{4}\right)$

"Phase gate"

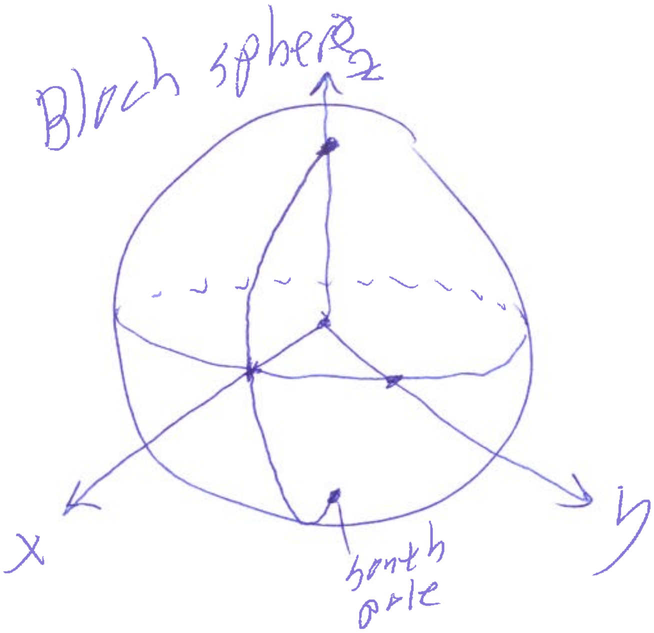
$$\forall \theta_1, \theta_2,\ R_x(\theta_1 + \theta_2) = R_x(\theta_1) R_x(\theta_2)$$

Fact: ~~Ever~~ For every 1-qubit unitary U, there exist unique $\varphi, \theta, \psi \in \mathbb{R}$ such that $\theta \in [0, \pi]$, $\varphi, \psi \in [0, 2\pi)$ such that

$$U \propto R_z(\varphi) R_y(\theta) R_z(\psi)$$

φ, θ, ψ are Euler angles

Geometrical "proof"



Any rotation (U) moves the North pole somewhere on the sphere → (θ, φ) spherical coords

2 steps:

1. Rotate about y-axis through angle θ $(0 \le \theta \le \pi)$ to move the north pole somewhere along the "positive" meridian ~~to~~ at the right latitude (colatitude)
   $R_y(\theta)$
2. Rotate around z-axis through angle φ to get the right longitude (latitude stays the same).
   $R_z(\varphi)$

Step 0: (before steps 1, 2):

Rotate about the z-axis through some arbitrary ~~angle~~ angle ψ [leaves the North pole fixed]

Rotating through ψ around an axis with spherical coords (θ, φ):

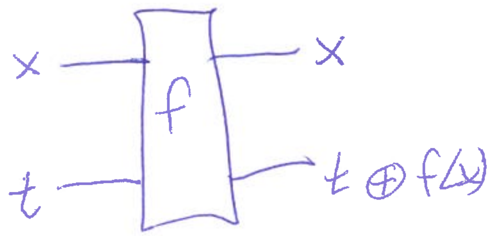
$$R_z(\varphi) R_y(\theta) R_z(\psi) R_y(-\theta) R_z(-\varphi)$$

"S^3 parameterization"

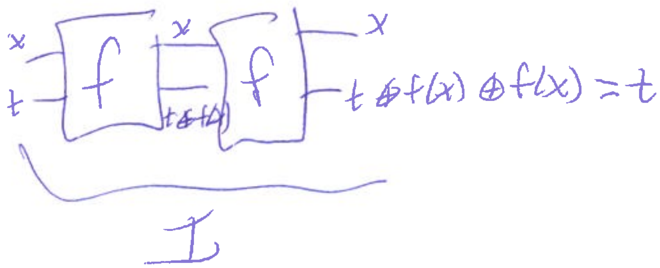
Deutsch's problem:

Given a Boolean function $f:\{0,1\} \to \{0,1\}$

Assume we have a quantum gate that implements f:



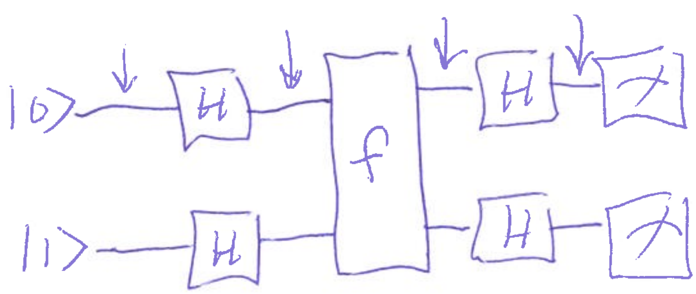
Note:



Want to determine whether f is constant or nonconstant

Classically, need 2 queries to f to decide this

But here is a quantum circuit that uses f <u>once</u> to decide it.



$$|01\rangle \xmapsto{H_1} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle = \frac{1}{\sqrt{2}}\cancel{(|01\rangle + |11\rangle)}$$

$$\xmapsto{H_2} \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)$$

$$= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$\bar{x} := x \oplus 1$$

$$\xmapsto{f} \frac{1}{2}\Big(|0, f(0)\rangle - |0, \overline{f(0)}\rangle + |1, f(1)\rangle - |1, \overline{f(1)}\rangle \Big)$$

$\cancel{H\otimes H}$ Note $\quad H|b\rangle = \dfrac{|0\rangle + (-1)^b |1\rangle}{\sqrt{2}} \quad b \in \{0,1\}$

$$\xmapsto{H_1} \frac{1}{2\sqrt{2}}\Big((|0\rangle + |1\rangle)\otimes |f(0)\rangle - (|0\rangle + |1\rangle)\otimes|\overline{f(0)}\rangle + (|0\rangle - |1\rangle)\otimes|f(1)\rangle - (|0\rangle - |1\rangle)\otimes|\overline{f(1)}\rangle \Big)$$

$$\xmapsto{H_2} \frac{1}{4}\Big((|0\rangle + |1\rangle)\otimes(|0\rangle + (-1)^{f(0)}|1\rangle) - (|0\rangle + |1\rangle)\otimes(|0\rangle + (-1)^{\overline{f(0)}}|1\rangle) + (|0\rangle - |1\rangle)\otimes(|0\rangle + (-1)^{f(1)}|1\rangle) - (|0\rangle - |1\rangle)\otimes(|0\rangle + (-1)^{\overline{f(1)}}|1\rangle) \Big)$$

f constant:
$f(x) = 0$, say

$$= \frac{1}{4}\Big((|0\rangle + |1\rangle)\otimes(|0\rangle + |1\rangle) - (|0\rangle + |1\rangle)\otimes(|0\rangle - |1\rangle) + (|0\rangle - |1\rangle)\otimes(|0\rangle + |1\rangle) - (|0\rangle - |1\rangle)\otimes(|0\rangle - |1\rangle) \Big)$$

$$= \frac{1}{4}\big(4|01\rangle\big) = |01\rangle$$

$\cancel{f(x)}$

$f(x)=1$ constant:

$$= \frac{1}{4}\Big((|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) - (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) + (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) - (|0\rangle - |1\rangle) \otimes (|0\rangle + |1\rangle) \Big)$$

$$= \frac{1}{4}\left(-4|01\rangle\right) = -|01\rangle$$

$f(x)=x$

$$= \frac{1}{4}\Big((+)(+) - (+)(-) + (-)(-) - (-)(+) \Big) = \frac{1}{4}\left(4|11\rangle\right) = |11\rangle$$

$f(x)=\bar{x}$: $\cancel{\pm|10\rangle}$ $-\cancel{+}|11\rangle$ [check!]

$f(x)=0$	$f(x)=1$	$f(x)=x$	$f(x)=\bar{x}$
$\|01\rangle$	$-\|01\rangle$	$\|11\rangle$	$-\|11\rangle$

1st qubit measurement ~~given~~ gives answer $\begin{cases} 0 & \text{if } f \text{ constant} \\ 1 & \text{if } f \text{ nonconstant} \end{cases}$