CSCE 785
10/5/2023

Change of Base

Quantum Teleportation (of one qubit)

C. of. B: Operator $A \in \mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ is a $\mathbb{C}$-space)

rep. as a matrix (say $n \times n$, $n = \dim(\mathcal{H})$)

Note all $1 \le i, j \le n$

$$[A]_{ij} = e_i^T A e_j = e_i^* A e_j = \underline{\langle e_i, Ae_j \rangle}$$

Let $\{u_1, \ldots, u_n\}$ be any orthonormal basis for $\mathcal{H}$.

Define the $n \times n$ matrix A' by

$$[A']_{ij} := u_i^* A u_j = \langle u_i, Au_j \rangle$$

Let $U \in \mathcal{L}(\mathcal{H})$ be the unique operator that maps e_i to u_i: $u_i = Ue_i$ (i'th column of U)

[U preserves the inner product, so U is <u>unitary</u>.]

$\forall i, j,\ 1 \le i, j \le n$:

$$[A']_{ij} = u_i^* A u_j = \langle u_i, Au_j \rangle = \langle Ue_i, AUe_j \rangle$$

$$= \langle e_i, \underline{U^* AU} e_j \rangle = [U^* AU]_{ij}$$ with resp. to standard basis

Thus $A' = U^*AU$ — $\underline{U^*AU}$ unitary conjugate of A

$$UA' = UU^*AU = AU$$

$$UA'U^* = AUU^* = A$$

$\therefore$ $A = UA'U^*$ unitary conjugate.

<u>Def</u>: $A, B \in \mathcal{L}(\mathcal{H})$. Say A & B are <u>unitarily conjugate</u> if $A = UBU^*$ for some unitary U.

<u>Idea</u>: This means A & B can be rep by the same matrix over (different) bases.

<u>Ex</u>: "unitarily conjugate" is an equivalence relation.

[I is unitary; If U is unitary, then U^* is unitary; if U, V unitary, then UV is unitary]

Almost <u>all</u> properties of operators in $\mathcal{L}(\mathcal{H})$ are invariant under unitary conjugation: (U unitary arbitrary)

$$\operatorname{tr} A = \operatorname{tr}(UAU^*)$$

$$\det A = \det(UAU^*)$$

$$UA^*U^* = (UAU^*)^*$$

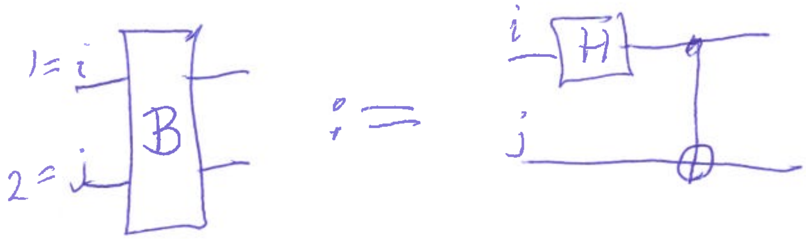
$$A = A^* \iff UAU^* = \cancel{UA^*U^*}\ (UAU^*)^*$$

$\therefore$ A is Hermitian iff UAU^* is Hermitian

Quantum Teleportation & Bell States

Four Bell states (all 2-qubit states, orth. basis for $(\mathbb{C}^2)^{\otimes 2} = \mathbb{C}^2 \otimes \mathbb{C}^2$)

Let



$H_i = \underbrace{I \otimes I \otimes \cdots \otimes I}_{i-1} \otimes H \otimes I \otimes \cdots \otimes$

So $\underline{B_{i,j}} = (C\text{-}X_{i,j})\, H_i$

B acting on qubits i & j

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$|\Phi^+\rangle \leftarrow B|00\rangle = (C\text{-}X_{1,2})\left(\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

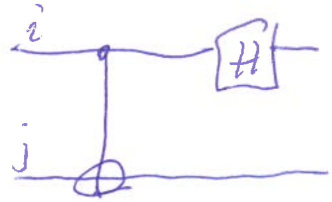
$|\Psi^+\rangle \leftarrow B|01\rangle = \cdots = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$

$|\Phi^-\rangle \leftarrow B|10\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

$|\Psi^-\rangle \leftarrow B|11\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

$$B^* = \left((C\text{-}X_{i,j})H_i\right)^* = H_i^* \, C\text{-}X_{i,j}^*$$

$$= H_i (C\text{-}X_{i,j}) =$$



Setup:

Alice has a qubit in some arbitrary state $|\psi\rangle$ (Alice might not know what $|\psi\rangle$ is)

She wants to transfer this state to Bob

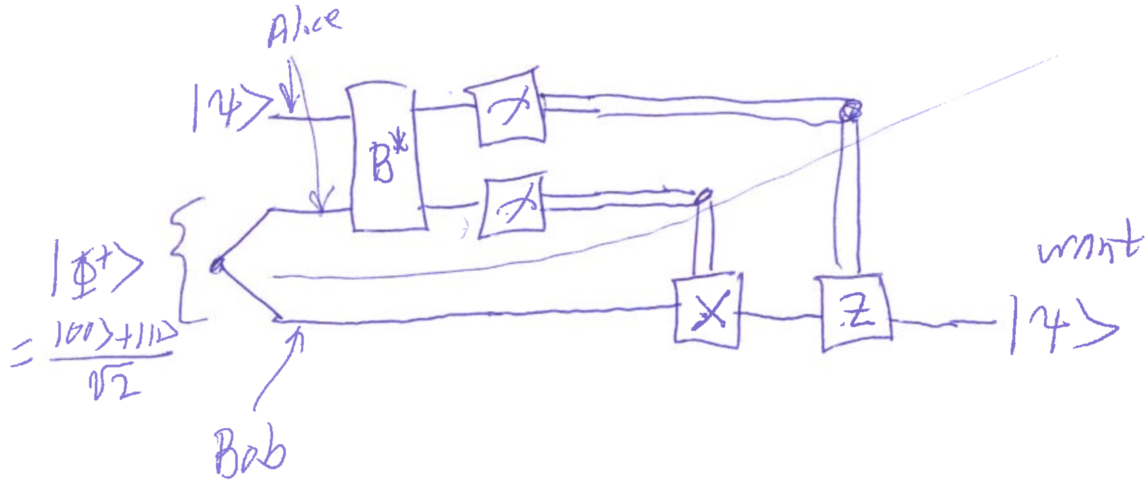
But she can only communicate with Bob classically. (Sharing classical data.)

This is provably impossible. BUT If Alice and Bob share a 2-qubit ~~state~~ register in state $\underbrace{|\Phi^+\rangle}_{\text{EPR pair}}$, then it is possible.

prior entanglement

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Teleportation circuit



$B^* = H_1(C\text{-}X_{1,2})$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad (\alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1)$

Entire state is $|\psi\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle) \otimes (|00\rangle + |11\rangle)$

$$= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

$$\xrightarrow{C\text{-}X_{1,2}} \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

$$\xrightarrow{H_1} \frac{1}{2}\Big(\alpha(|0\rangle + |1\rangle)|00\rangle + \alpha(|0\rangle + |1\rangle)|11\rangle + \beta(|0\rangle - |1\rangle)|10\rangle + \beta(|0\rangle - |1\rangle)|01\rangle\Big)$$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$= \frac{1}{2}\Big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big)$$

Case	Alice		Bob:		
	00	$\Longrightarrow$	$\alpha|0\rangle + \beta|1\rangle$	$\xmapsto{I}$	$\alpha|0\rangle + \beta|1\rangle = |\psi\rangle$
prob	01	$\Longrightarrow$	$\alpha|1\rangle + \beta|0\rangle$	$\xmapsto{X}$	$\alpha|0\rangle + \beta|1\rangle$ //
$\frac{1}{4}$ for	10		$\alpha|0\rangle - \beta|1\rangle$	$\xmapsto{Z}$	$\alpha|0\rangle + \beta|1\rangle$
each	11		$\alpha|1\rangle - \beta|0\rangle$	$\xmapsto{ZX}$	$\alpha|0\rangle + \beta|1\rangle$ //