CSCE 785
10/3/2023

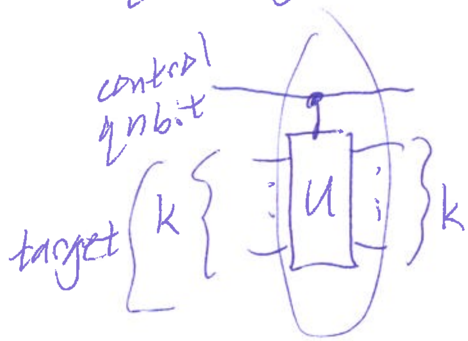
Install Qiskit (recommended)

Quantum Circuits (cont.):
controlled gates
measurement gates
quantum ≥ classical

controlled gates

Given k-qubit gate U (unitary) $(k \geq 1)$

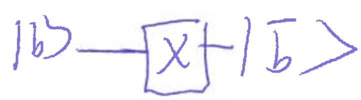
Have a $(k+1)$-qubit gate C-U (Controlled-U):

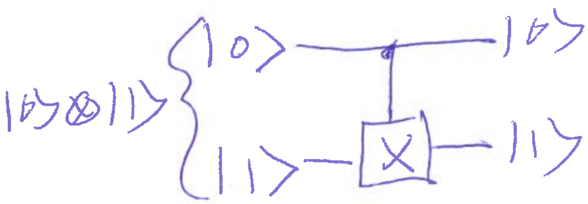


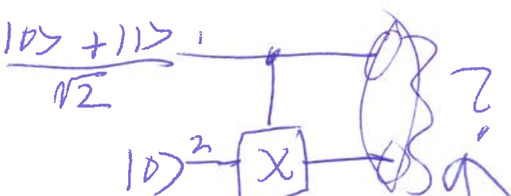
If control is in state $|0\rangle$, then nothing happens (to anything)

If control is in state $|1\rangle$, then U is applied to the target qubits (control stays $|1\rangle$)

Ex: Controlled-NOT (CNOT) (C-X)



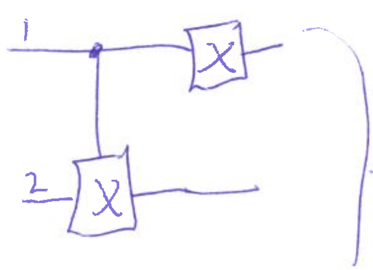




Input: $\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle$

$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

$\xmapsto{C\text{-}X_{1,2}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



. . . $\xmapsto{X_1} \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$

$$C\text{-}X_{1,2} = \begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & 1 & 0 & 0 & 0 \\ |01\rangle & 0 & 1 & 0 & 0 \\ |10\rangle & 0 & 0 & 0 & 1 \\ |11\rangle & 0 & 0 & 1 & 0 \end{array} \quad \begin{bmatrix} \text{state} \\ \text{vector} \end{bmatrix}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & X \end{array}\right] = \boxed{|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X}$$

$$|\varphi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\xmapsto{C\text{-}X_{1,2}} (C\text{-}X_{1,2})|\varphi\rangle = \left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X\right)|\varphi\rangle$$

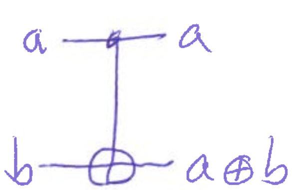
$$= \left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X\right)\alpha_{00}|00\rangle$$
$$+ \left(\quad \text{''} \quad \right)\alpha_{01}|01\rangle$$
$$+ \left(\quad \text{''} \quad \right)\alpha_{10}|10\rangle$$
$$+ \left(\quad \text{''} \quad \right)\alpha_{11}|11\rangle$$

$$= \alpha_{00}\left(|0\rangle\overbrace{\langle 0|0\rangle}^{1} \otimes I|0\rangle + |1\rangle\underbrace{\langle 1|0\rangle}_{} \otimes X|0\rangle\right)$$
$$= \alpha_{00}\left(|0\rangle \otimes |0\rangle\right)$$
$$= \alpha_{00}|00\rangle$$

<u>Convention</u>

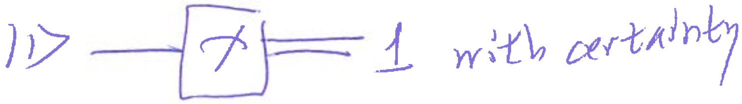
This is a classical gate

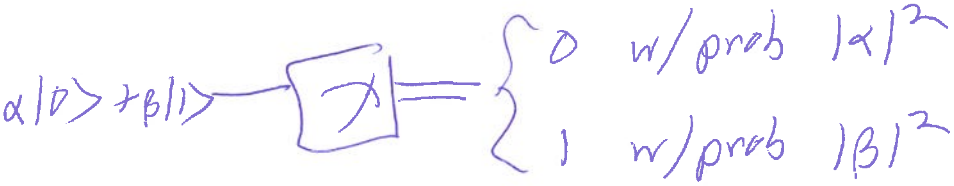
$a, b \in \{0, 1\}$

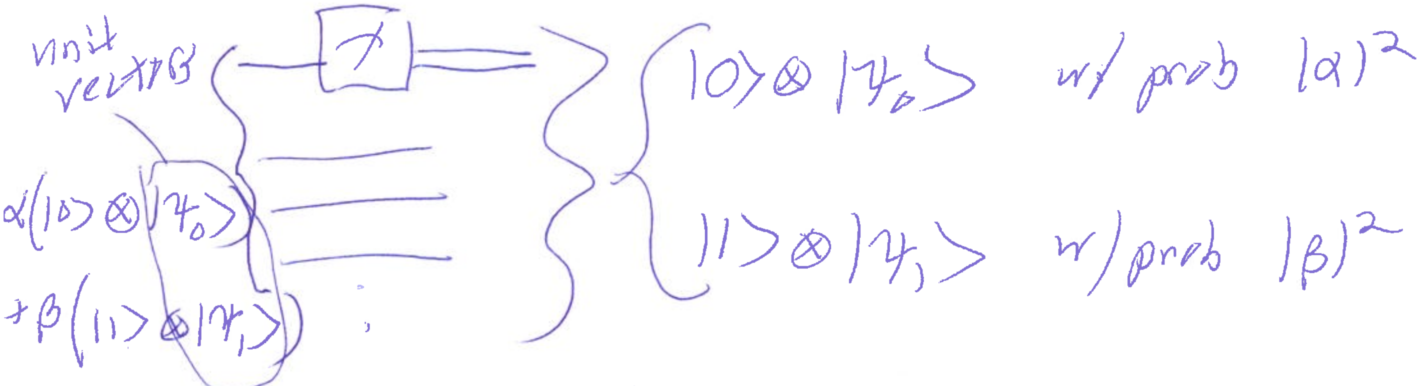


$(a+b) \bmod 2 = a \text{ XOR } b$

Measurement Gates





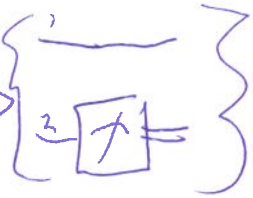


"state collapse"

Algebraically,  corresponds to the projective measurement with projectors

$$P_0 := |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_1 := I - P_0 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\alpha_{00}|00\rangle + \dots + \alpha_{11}|11\rangle$ 

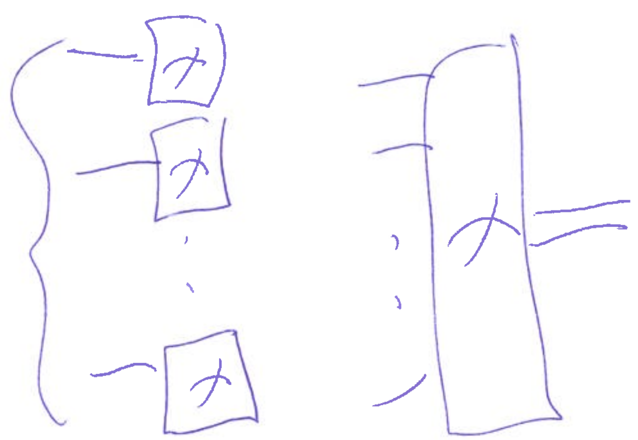
$$\begin{cases} \dfrac{\alpha_{00}|0\rangle + \alpha_{10}|1\rangle}{\sqrt{|\alpha_{00}|^2+|\alpha_{10}|^2}} \otimes |0\rangle & \text{w/prob } |\alpha_{00}|^2+|\alpha_{10}|^2 \\[2ex] \dfrac{\alpha_{01}|0\rangle + \alpha_{11}|1\rangle}{\sqrt{|\alpha_{01}|^2+|\alpha_{11}|^2}} \otimes |1\rangle & \text{w/prob } |\alpha_{01}|^2+|\alpha_{11}|^2 \end{cases}$$

$$= (\alpha_{00}|0\rangle + \alpha_{10}|1\rangle) \otimes |0\rangle + (\alpha_{01}|0\rangle + \alpha_{11}|1\rangle) \otimes |1\rangle$$

$$= \boxed{\sqrt{|\alpha_{00}|^2+|\alpha_{10}|^2}} \left(\frac{\alpha_{00}|0\rangle + \alpha_{10}|1\rangle}{\sqrt{|\alpha_{00}|^2+|\alpha_{10}|^2}} \right) \otimes |0\rangle$$

$$+ \sqrt{|\alpha_{01}|^2+|\alpha_{11}|^2} \left(\frac{\alpha_{01}|0\rangle + \alpha_{11}|1\rangle}{\sqrt{|\alpha_{01}|^2+|\alpha_{11}|^2}} \right) \otimes |1\rangle$$

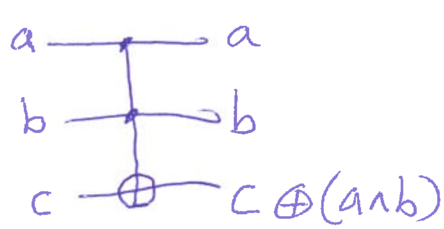
k qubits



$x \in \{0,1\}^k$ with prob ...

Quantum vs Classical comp.

Toffoli gate (classical gate)



Controlled CNOT

C-C-X

Every classical deterministic computation can be carried out by a Boolean circuit with $\vee$ (OR), $\wedge$ (AND) gates on 2 inputs an a NOT ($\neg$) gate on one input

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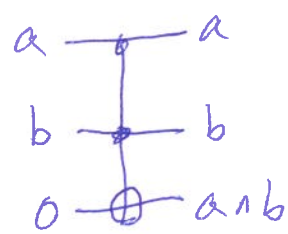
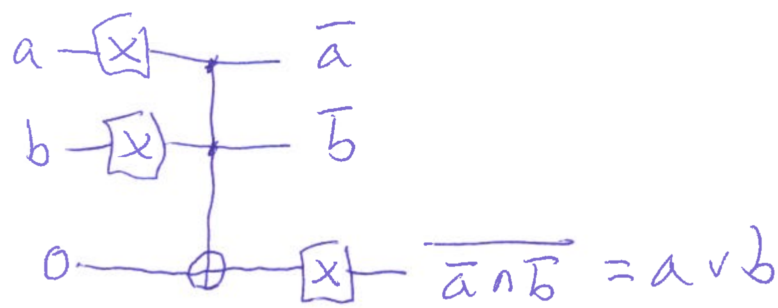
Simulating a classical computation with a quantum circuit:

AND-gate



OR-gate



NOT-gate

Copy primitive