CSCE 785
9/28/2023

Tensor Products (cont.)
Quantum registers
Quantum Circuits

1

<u>Def</u>: Let V, W be $\mathbb{C}$-spaces. Define $\mathbb{C}$-space $V \otimes W := \operatorname{span}\{v \otimes w : v \in V, w \in W\}$

$$\left.\begin{array}{l} n := \dim(V) \\ r := \dim(W) \end{array}\right\} \Rightarrow \dim(V \otimes W) = nr$$

Inner product on $V \otimes W$: $\quad v_1, v_2 \in V$, $\quad w_1, w_2 \in W$

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle_V \cdot \langle w_1, w_2 \rangle_W$$

Then if $\{v_1, \ldots, v_n\}$ is an ortho. basis for V
$\{w_1, \ldots, w_r\}$ . . . . W:

Then $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le r\}$ is an ortho basis for $V \otimes W$.
A (~~the~~) product basis for $V \otimes W$

Verify $\langle v_{i_1} \otimes w_{j_1}, v_{i_2} \otimes w_{j_2} \rangle$

$$= \langle v_{i_1}, v_{i_2} \rangle \langle w_{j_1}, w_{j_2} \rangle = \delta_{i_1 i_2} \delta_{j_1 j_2} = \begin{cases} 1 & \text{if } i_1 = i_2 \ \& \ j_1 = j_2 \\ 0 & \text{o.w.} \end{cases}$$

$(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$ (as $\mathbb{C}$-spaces) ②

(drop parens: $V_1 \otimes V_2 \otimes V_3$)

On ~~on~~ matrices (vectors, operators, etc) — forget last time

$\otimes$ is bilinear

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

$$(B + C) \otimes A = B \otimes A + C \otimes A$$

For $a \in \mathbb{C}$ (scalar)

$$(aA) \otimes B = a(A \otimes B) = A \otimes (aB)$$

Def: An n-qubit quantum register is a physical system with $\mathbb{C}$-space $(\mathbb{C}^2)^{\otimes n}$ basis $\{|0\rangle, |1\rangle\}$

$$\left[(\mathbb{C}^2)^{\otimes n} := \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} \right]$$

dimension 2^n

with product basis is $\{|b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle :$ where each $b_j \in \{0,1\}$ (label) $\}$ size 2^n

shorthands:

$$|b_1\rangle \otimes \cdots \otimes |b_n\rangle := |b_1\rangle|b_2\rangle\cdots|b_n\rangle$$
$$:= |b_1 b_2 \cdots b_n\rangle$$

so this basis is $\{|x\rangle : x \in \{0,1\}^n\}$

$$|0\rangle^{\otimes n} = |0\rangle|0\rangle\cdots|0\rangle = |0^n\rangle$$

standard basis for $(\mathbb{C}^2)^{\otimes n}$, the <u>computational basis</u>

<u>Example:</u> 2 qubits

$$\underbrace{|00\rangle}_{0} = |0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot\begin{bmatrix}1\\0\end{bmatrix}\\ \hline 0\cdot\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = e_1$$

$$\underbrace{|01\rangle}_{1} = |0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\cdot\begin{bmatrix}0\\1\end{bmatrix}\\ \hline 0\cdot\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = e_2$$

$$\underbrace{|10\rangle}_{2} = \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\cdot\begin{bmatrix}1\\0\end{bmatrix}\\ \hline 1\cdot\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = e_3$$

$$\underset{3}{|11\rangle} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = e_4$$

General 2-qubit unit vector is

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$= \sum_{x\in\{0,1\}^2} \alpha_x |x\rangle$$

$$1 = \langle\psi|\psi\rangle = \cancel{\langle \sum_x \alpha_x |x\rangle} \left(\sum_x \alpha_x |x\rangle\right)^* \left(\sum_y \alpha_y |y\rangle\right)$$

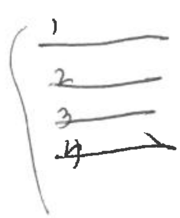
$$= \left(\sum_x \alpha_x^* \langle x|\right)\left(\sum_y \alpha_y |y\rangle\right)$$

$$= \sum_{x,y} \alpha_x^* \alpha_y \langle x|y\rangle = \sum_x \alpha_x^* \alpha_x = \sum_x |\alpha_x|^2$$

Quantum Circuit Model:

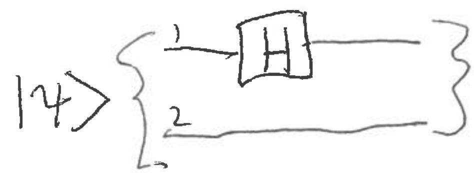
(one or more) Quantum registers depicted as horizontal wires

E.g., 4-qubit register



Initial state is given on the left, gates (usually unitary operators) are applied left to right, i.e., time moves from left to right.

Ex: 2-qubit reg.



$\underbrace{H_1}_{}$ — apply H to 1st qubit

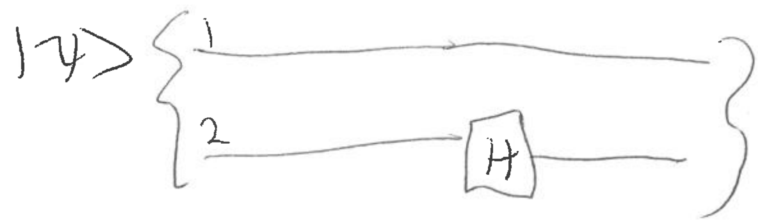
$$|\psi\rangle = \overbrace{(H \otimes I)}^{H_1} |\psi\rangle$$

For unitary operators: $\rho = |\psi\rangle\langle\psi|$ pure state

$$\xmapsto[]{\substack{\text{unitary}\\ U}} U\rho U^* = U|\psi\rangle\langle\psi|U^*$$

$$= (U|\psi\rangle)(U|\psi\rangle)^*$$

So only need to retain $U|\psi\rangle$



$$H_2|\psi\rangle = (I \otimes H)|\psi\rangle$$

$$H_1|00\rangle = (H \otimes I)(|0\rangle \otimes |0\rangle) = (H|0\rangle) \otimes (I|0\rangle)$$

$$= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$H_1|11\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |11\rangle\right)$$

$$H_1|10\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right) \quad \text{etc.}$$

$$H_1|01\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$$

$$H_2|00\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$$

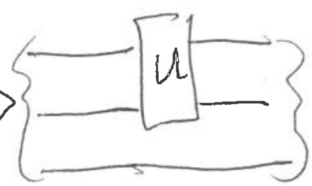
etc.

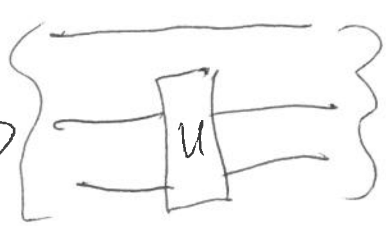
Seen how H_1 acts on comp. basis vectors.

Determines how H_1 acts on arbitrary vectors in $(\mathbb{C}^2)^{\otimes 2} \cong \mathbb{C}^4$

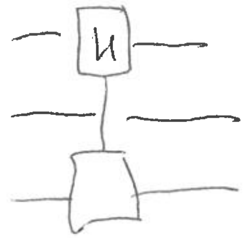
<u>Multiqubit gates</u>: acts on 2 or more qubits

U is a 2-qubit unitary operator

$|\psi\rangle$  $U_{1,2}|\psi\rangle = (U \otimes \underbrace{I}_{\text{1-qubit}})|\psi\rangle$

$|\psi\rangle$  $U_{2,3}|\psi\rangle = (I \otimes U)|\psi\rangle$

$U_{1,3}$?



$= \cancel{(I_2 \otimes U)|\psi\rangle}$

Swap operator (2-qubit operator)

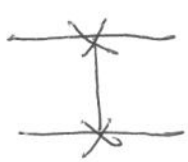
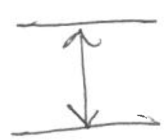
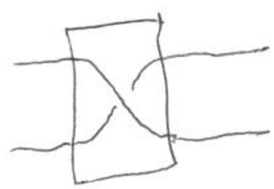
$$\text{SWAP} = \begin{array}{c} \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\left.\begin{aligned} \text{SWAP}|00\rangle &= |00\rangle \\ \text{SWAP}|01\rangle &= |10\rangle \\ \text{SWAP}|10\rangle &= |01\rangle \\ \text{SWAP}|11\rangle &= |11\rangle \end{aligned}\right)$$

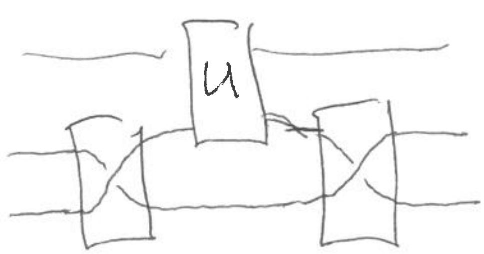
Can show: any 1-qubit states $|\psi\rangle, |\phi\rangle$,

$$\text{SWAP}\left(|\psi\rangle \otimes |\phi\rangle\right) = |\phi\rangle \otimes |\psi\rangle$$

As a quantum gate in a circuit:

 or  or 

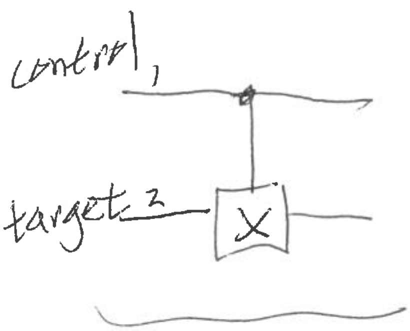
$$U_{1,3} = \text{SWAP}_{2,3}\, U_{1,2}\, \text{SWAP}_{2,3} = \text{SWAP}_{1,2}\, U_{2,3}\, \text{SWAP}_{1,2}$$



CNOT operator (controlled NOT), or
C-X (2-qubits)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ Pauli gate}$$

$$\left.\begin{aligned} X|0\rangle &= |1\rangle \\ X|1\rangle &= |0\rangle \end{aligned}\right\}$$ X is the logical ~~not~~ NOT operator on a single qubit



$C\text{-}X_{1,2}$