CSCE 785
9/26/23

1-qubit unitaries & the Bloch Sphere
Change of (orthonormal) basis
Tensor Products

Seen that 1-qubit states correspond to points on the unit sphere (Bloch sphere) in $\mathbb{R}^3$

1-qubit unit unitaries correspond to rotations in $\mathbb{R}^3$.

Recall: 1-qubit state $\rho = \frac{1}{2}(I + \boxed{a_1 X + a_2 Y + a_3 Z})$

$\sigma = \frac{1}{2}(I + \boxed{b_1 X + b_2 Y + b_3 Z})$

$\langle \rho, \sigma \rangle = ?$

$a_i \in \mathbb{R}$ $\quad a_1^2 + a_2^2 + a_3^2 = 1$

$b_i \in \mathbb{R}$ $\quad b_1^2 + b_2^2 + b_3^2 = 1$

$$\langle \overbrace{a_1 X + a_2 Y + a_3 Z}^{\text{Hermitian}}, b_1 X + b_2 Y + b_3 Z \rangle$$

$$= tr\left((a_1 X + a_2 Y + a_3 Z)(b_1 X + b_2 Y + b_3 Z)\right)$$

$$= tr\left((a_1 b_1 + a_2 b_2 + a_3 b_3) I + (\text{terms with zero trace})\right)$$

$$= 2(\vec{a} \cdot \vec{b})$$

Follows that $\langle \rho, \sigma \rangle = \frac{1 + \vec{a} \cdot \vec{b}}{2}$

U — 1-qubit unitary

$$\rho \overset{U}{\longmapsto} U\rho U^* = \rho'$$

$$\sigma \overset{U}{\longmapsto} U\sigma U^* = \sigma'$$

$$\underline{\langle \rho', \sigma' \rangle} = tr\left((\rho')^* \sigma'\right) = tr(\rho'\sigma')$$

$$= tr\left(U\rho \cancel{U^* U} \sigma U^*\right)$$

$$= tr\left(U\rho\sigma U^*\right)$$

$$= tr\left(U^* U \rho\sigma\right) = tr(\rho\sigma) = \underline{\langle \rho, \sigma \rangle}$$

conj. by U preserves $\langle \rho, \sigma \rangle$, so it also preserves $\vec{a} \cdot \vec{b}$.

Pauli operators $\quad I, \quad X, \quad Y, \quad Z$

$$X\rho X^* = X\rho X = \frac{1}{2} X(I + a_1 X + a_2 Y + a_3 Z) X$$

$$= \frac{1}{2}\left(I + a_1 X - a_2 Y - a_3 Z\right)$$

point $(a_1, -a_2, -a_3)$

X – rotation about x-axis by π ($= 180°$)

Y – " " y-axis " π

Z – " " z-axis " π

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$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ Hadamard operator

H is Hermitian & unitary (like X, Y, Z)

$$H = \frac{1}{\sqrt{2}}(X + Z) \qquad \boxed{H^2 = I}$$

$$HXH = \cancel{\tfrac{1}{\sqrt{2}}(HXH} \; \frac{1}{\sqrt{2}}(X+Z)\,X\,\frac{1}{\sqrt{2}}(X+Z)$$

$$= \frac{1}{2}(XXX + XXZ + ZXX + ZXZ)$$

$$= \frac{1}{2}(\cancel{X} + Z + Z \cancel{-X})$$

$$= Z$$

Ex: $HZH = X$

$HYH = -Y$

$$H\rho H = \frac{1}{2}(HIH + a_1 HXH + a_2 HYH + a_3 HZH)$$

$$= \frac{1}{2}(I + a_1 Z - a_2 Y + a_3 X)$$

$$= \frac{1}{2}(I + a_3 X - a_2 Y + a_1 Z)$$

corresp. to $(a_3, -a_2, a_1)$

H corresp to a π-rotation about the axis $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$\left[H = \frac{1}{\sqrt{2}}(X + Z)\right]$$

Let $A := u_1 X + u_2 Y + u_3 Z$

where $u_1, u_2, u_3 \in \mathbb{R}$ & $u_1^2 + u_2^2 + u_3^2 = 1$

Then A is Hermitian & unitary & conjugation by A corresponds to a $180°$ rotation about the axis (u_1, u_2, u_3)

Two more 1-qubit ops:

$$S := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad \left[S^2 = Z\right]$$

chk that S is unitary $S^*S = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$SXS^* = Y$

$SYS^* = -X$

$SZS^* = Z \quad \left(= SSSS^* = ZSS^* = Z\right)$

S is a $\frac{\pi}{2}$-rotation counterclockwise about the z-axis

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$ "$\frac{\pi}{8}$-gate"

$$\propto \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}$$

$$\boxed{\begin{aligned} T^2 &= S \\ T^4 &= Z \end{aligned}}$$

phase factor

$U \propto V$ means $U = \boxed{e^{i\theta}} V$ for some $\theta \in \mathbb{R}$

"U, V are equal up to an overall phase factor"
(global)

$$U \rho U^* = (e^{i\theta} V) \rho (e^{i\theta} V)^*$$
$$= e^{i\theta} V \rho e^{-i\theta} V^* = V \rho V^*$$

$$T X T^* = \frac{1}{\sqrt{2}}(X + Y)$$

$$T Y T^* = \frac{1}{\sqrt{2}}(-X + Y)$$

$$T Z T^* = Z$$

T gives rotation about the z-axis through $\frac{\pi}{4}$

Generally, $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$ gives a rot. about the z-axis through ~~ate~~ angle θ

Tensor Products

Def: Let A, B be any two matrices. Define

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & \cdot \\ a_{21}B & a_{22}B & \cdot & \cdot \\ \vdots & \vdots & \ddots & \\ & \vdots & & \end{bmatrix}$$

where $a_{ij} := [A]_{ij}$

Tensor product (Kronecker product, direct product, outer product) of A & B.

Useful properties:

0) If A is $m \times n$ and B is $r \times s$, then $A \otimes B$ is $mr \times ns$

1) $a \otimes b = ab$ ($a, b \in \mathbb{C}$ scalars)

2) More generally, if A is $m \times 1$ and B is $1 \times n$ then $A \otimes B = AB$

(*) 3) $(A \otimes B)(C \otimes D)$

$$= AC \otimes BD$$

[LHS is conformant $\Longleftarrow$ RHS is conformant]

4) $(A \otimes B)^* = A^* \otimes B^*$

5) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

(associativity)

6) $\operatorname{tr}(A \otimes B) = (\operatorname{tr} A)(\operatorname{tr} B)$

$\therefore \langle A \otimes B, C \otimes D \rangle = \operatorname{tr}\left((A \otimes B)^*(C \otimes D)\right)$

$= \operatorname{tr}\left((A^* \otimes B^*)(C \otimes D)\right) = \operatorname{tr}\left(A^*C \otimes B^*D\right)$

$= \left(\operatorname{tr}(A^*C)\right)\left(\operatorname{tr}(B^*D)\right) =$

$\langle A, C \rangle \langle B, D \rangle$