CSCE 785
9/21/2023

Projective Measurements

Classical info from a quantum system

Recall: $\mathcal{H}$ $\mathbb{C}$-space. $P \in \mathcal{L}(\mathcal{H})$ is a (orthogonal) projector if $PP = P$ & $P = P^*$ $[\Leftrightarrow P = P^*P]$

Def: A complete set of orthogonal projectors (csop) is a collection $\{P_1, \ldots, P_k\}$ of nonzero projectors on $\mathcal{H}$ such that $\sum_{j=1}^{k} P_j = I$.

Lemma: Let $P_1, \ldots, P_k \in \mathcal{L}(\mathcal{H})$ be projectors. If $\sum_{j=1}^{k} P_j$ is a projector, then $P_i P_j = 0$ for all $1 \le i, j \le k$, ~~~~ $i \ne j$.

Proof: Let $P := \sum_{j=1}^{k} P_j$

By assumption

$$P = PP = \left(\sum_j P_j\right)\left(\sum_\ell P_\ell\right) = \sum_{j,\ell} P_j P_\ell = \sum_j P_j + \sum_{j \ne \ell} P_j P_\ell$$

$$\parallel$$

$$\sum_j P_j$$

so $0 = \sum'_{j \neq l} P_j^* P_l$ Trace of both sides:

$$0 = \operatorname{tr} 0 = \sum'_{j \neq l} \operatorname{tr}(P_j^* P_l) = \sum'_{j \neq l} \underbrace{\langle P_j, P_l \rangle}_{\geq 0}$$

take this on faith for now $\left\{ \begin{array}{l} \geq 0 \\ = 0 \text{ iff } P_j P_l = 0 \end{array} \right.$

$\cancel{= \sum_{j \neq l} \cdots}$

$$\Rightarrow \forall j \neq l, \ \langle P_j, P_l \rangle = 0 \quad \therefore P_j P_l = 0 \quad \blacksquare$$

A projective measurement corresponds with a csop.
Let Ω be a finite set (Ω sample space of outcomes). $\Omega = \{1, \dots, k\}$, each j corresp. to P_j in the csop $\{P_1, \dots, P_k\}$.

Given a quantum state $\rho \in \mathcal{L}(\mathcal{H})$
Performing this measurement on the system in state ρ yields outcome $j \in \Omega$ with probability $\boxed{\operatorname{tr}(P_j \rho) = \underbrace{\langle P_j, \rho \rangle}_{\geq 0}}$ $[P_j = P_j^*]$

take on faith for now

Check that probabilities add to 1:

$$\sum_{j=1}^{k} \langle P_j, \rho \rangle = \left\langle \sum_j P_j, \rho \right\rangle = \langle I, \rho \rangle$$

$$= tr(I\rho) = tr(\rho) = 1$$

If $\rho = |\psi\rangle\langle\psi|$ is a pure state, then

$$tr\,\rho = tr(|\psi\rangle\langle\psi|) = tr\langle\psi|\psi\rangle$$

$$= \langle\psi|\psi\rangle = \||\psi\rangle\|^2 = 1$$

Without Dirac notation

$\rho = uu^*$ for some unit (column) vector $u \in \mathcal{H}$.

$$tr\,\rho = tr(uu^*) = tr(u^*u) = u^*u = \langle u, u\rangle$$

$$= \|u\|^2 = 1$$

Given outcome j as the the measurement result, then the post-measurement state is

$$\boxed{\rho_j = \frac{P_j \rho P_j}{\Pr\{j\}} = \frac{P_j \rho P_j}{\langle P_j, \rho\rangle}}$$

Special case: ρ is a pure state $\rho = |\psi\rangle\langle\psi|$

$$\Pr[j] = \langle P_j, \rho \rangle = tr(P_j \rho) = tr(P_j |\psi\rangle\langle\psi|)$$

$$= tr(\underbrace{\langle\psi| P_j |\psi\rangle}_{\substack{\text{scalar} \\ = 1\times 1 \text{ matrix}}}) = \boxed{\langle\psi| P_j |\psi\rangle}$$

Post-measurement state given outcome j is

$$\frac{P_j \rho P_j}{\Pr[j]} = \frac{1}{\Pr[j]} P_j |\psi\rangle\langle\psi| P_j = \frac{P_j |\psi\rangle\langle\psi| P_j^*}{P[j]}$$

$$= \frac{P_j |\psi\rangle (P_j |\psi\rangle)^*}{\Pr[j]} = |\varphi\rangle\langle\varphi| \quad \text{where}$$

$$|\varphi\rangle := \frac{P_j |\psi\rangle}{\sqrt{\Pr[j]}} = \frac{P_j |\psi\rangle}{\sqrt{\langle\psi| P_j |\psi\rangle}}$$

Check that $\langle\varphi|\varphi\rangle = 1$:

$$\langle\varphi|\varphi\rangle = \frac{1}{\Pr[j]} (\langle\psi| P_j)(P_j |\psi\rangle)$$

$$= \frac{1}{\Pr[j]} \langle\psi| P_j P_j |\psi\rangle = \frac{1}{\Pr[j]} \langle\psi| P_j |\psi\rangle$$

$$= \frac{\Pr\{j\}}{\Pr\{j\}} = 1. \checkmark$$

$AB^* = 0$

~~$A^*AB^*B = 0$~~

Lemma: Let $A, B \in \mathcal{L}(\mathcal{H})$. Then $\langle A^*A, B^*B \rangle \geq 0$ with equality holding iff ~~$AB = 0$~~.

Proof: $\langle A^*A, B^*B \rangle = tr(A^*AB^*B) = tr(BA^*AB^*)$

$= tr((AB^*)^*AB^*) = \langle AB^*, AB^* \rangle = \|AB^*\|_2 \geq 0$

with equality iff $AB^* = 0$.

If $AB^* = 0$ then $A^*(AB^*)B = 0$

$\therefore = 0$ implies $A^*AB^*B = 0$. ▨

~~Applying this for pure state ρ:~~ Applying to $\langle P_j, P_\ell \rangle$

~~$\langle P_j, \rho \rangle$~~ $\quad \langle P_j, P_\ell \rangle = \langle P_j^* P_j, P_\ell^* P_\ell \rangle$

Applying to $\langle P_j, \rho \rangle$ where

$\rho = uu^*$ $\quad \boxed{P_j = P_j^* P_j}$

$\boxed{\rho = V^*V}$ where $V := u^*$

≥ 0 with $=$

iff $P_j P_\ell^* = 0$

iff $P_j P_\ell = 0$

[Lemma applies for nonsquare matrices also]

1-qubit measurements:

measuring spin in the z-direction:

$$\left\{ \overset{\uparrow}{\boxed{\frac{I+Z}{2}}}, \overset{\downarrow}{\boxed{\frac{I-Z}{2}}} \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Pauli z-matrix

$$\rho = \frac{1}{2}(I + a_1 X + a_2 Y + a_3 Z)$$

$a_1, a_2, a_3 \in \mathbb{R}$ with $a_1^2 + a_2^2 + a_3^2 = 1$.

$$\Pr[\uparrow] = \left\langle \frac{I+Z}{2}, \rho \right\rangle = \frac{1}{4}\langle I+Z, I + a_1 X + a_2 Y + a_3 Z \rangle$$

$$= \frac{1}{4}\left(\underbrace{\langle I, 2\rho \rangle}_{=tr(2\rho)=2} + \underbrace{\langle Z, 2\rho \rangle}_{2a_3} \right) = \frac{1 + a_3}{2}$$

$= \dfrac{1 + \hat{s}\cdot\hat{z}}{2}$ where $\hat{s} \in \mathbb{R}^3$ is the spin direction (unit vector)

$\hat{z} \in \mathbb{R}^3$ is the unit vector on the z-axis.