CSCE 785
9/19/2023

Finish char of 1-qubit state
Axioms of QM
Pauli matrices
Dirac Notation

From last time

$$uu^* = \begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & e^{-i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Double angle fmlas

$$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin(2x) = 2\sin x \cos x$$

$$\therefore uu^* = \frac{1}{2}\begin{bmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & (\cos\varphi - i\sin\varphi)\sin\theta \\ (\cos\varphi + i\sin\varphi)\sin\theta & 1-\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\left(I + \cos\varphi\sin\theta\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \sin\varphi\sin\theta\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \cos\theta\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(I + \overset{a_1}{\overset{\|}{a_x}} X + \overset{a_2}{\overset{\|}{a_y}} Y + \overset{a_3}{\overset{\|}{a_z}} Z\right) = \frac{1}{2}\left(I + \sum_{i=1}^{3} a_i \sigma_i\right)$$

where

$$I = \overset{\sigma_0}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \quad X = \overset{\sigma_1}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \quad Y = \overset{\sigma_2}{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}, \quad Z = \overset{\sigma_3}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

and $(a_x, a_y, a_z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

= cartesian coords of the spin!

Pauli (spin) matrices.

Properties of the Pauli matrices:

1. $\operatorname{tr} I = 2$, $\operatorname{tr} X = \operatorname{tr} Y = \operatorname{tr} Z = 0$
2. $\forall\, 0 \le i, j \le 3 \quad \langle \sigma_i, \sigma_j \rangle = 2\delta_{ij}$
3. $\boxed{I^2 = X^2 = Y^2 = Z^2 = I}$

   i.e. I, X, Y, Z are all Hermitian $(A^* = A)$ and unitary $(U^*U = I)$
4. $$\boxed{\begin{aligned} XY &= -YX = iZ \\ YZ &= -ZY = iX \\ ZX &= -XZ = iY \end{aligned}}$$

$\langle A, B \rangle = \operatorname{tr}(A^* B)$

More QM axioms

<u>Recall</u>: $\mathcal{H}$ is a $\mathbb{C}$-space, corresponds to a physical system. A <u>pure state</u> of $\mathcal{H}$ ("the system") is any 1-dim projector $\rho = uu^*$ (any unit vector u)

<u>Axiom</u> (time evolution of the state of a physical system). ~~For~~ Any physical system isolated from its environment evolves <u>unitarily</u>. That means, letting $\mathcal{H}$ be the $\mathbb{C}$-space of the system and any time coordinates $t_1, t_2 \in \mathbb{R}$, there is a unitary operator $U = U(t_1, t_2) \in \mathcal{L}(\mathcal{H})$ such that, if $\rho \in \mathcal{L}(\mathcal{H})$ is the state of the system at time t_1, then then the state of the system at time t_2 is

$$U\rho U^* \quad \left[= U(t_1, t_2)\,\rho\, U(t_1, t_2)^*\right].$$

Sanity Check: ~~Let~~ Let $\rho = uu^*$ (some unit $u \in \mathcal{H}$)

Then $U\rho U^* = Uuu^*U^* = Uu(Uu)^*$

u is a unit vector $(\langle u,u\rangle = 1 = \|u\|^2)$

U is unitary: $\langle Uu, Uu\rangle = \langle u, \underbrace{U^*U}_{I} u\rangle = \langle u,u\rangle = 1$

$\therefore$ Uu is a unit vector

$\therefore$ $(Uu)(Uu)^*$ is a 1-dim projector.

Add'l props

moreover

$U(t,t) = I$ $\quad [\, I\rho I^* = I\rho I = \rho \,]$

$U(t_2,t_1) = U(t_1,t_2)^*$

$U(t_2,t_3)\,U(t_1,t_2) = U(t_1,t_3)$

Later: $U(t_1,t_2) = e^{iH(t_1-t_2)}$ for some $H \in \mathcal{L}(\mathcal{H})$
in many cases

Dirac Notation. $\mathcal{H}$ $\mathbb{C}$-space

— A unit (column) vector in $\mathcal{H}$ ~~can be~~ is written as $|\psi\rangle$ (a "ket"). Here ψ is some label that identifies the vector. E.g. $|0\rangle, |1\rangle, |\uparrow\rangle, |\downarrow\rangle$, etc.

[some people identify $u = |u\rangle$. We won't]

Let $|\psi\rangle$ be a ket.

Its adjoint (a row vector) is

$$|\psi\rangle^* =: \langle\psi| \quad (\text{a "bra"})$$

$|\varphi\rangle, |\psi\rangle$ kets. Inner product is

$$(|\varphi\rangle)^* |\psi\rangle = \langle\varphi|\psi\rangle \quad (\text{"bra-ket" bracket})$$

Also: $|\psi\rangle\langle\varphi| \in \mathcal{L}(\mathcal{H})$ ("ket-bra")

$|\varphi\rangle\langle\varphi|$ is a one-dim projector ("bow tie")

Operators in Dirac notation: uppercase Roman letters

but, for $A \in \mathcal{L}(\mathcal{H})$

$A^\dagger$ ("A-dagger") denotes the adjoint (we use A^*)

$\| |\varphi\rangle \|$?