CSCE 785
9/14/2023

QM fundamentals
Qubits

Recall: electron spin is modeled by a 2-dim $\mathbb{C}$-space $\mathbb{C}^2$.

Arbitrarily:

$$\left.\begin{array}{l}\text{spin up } |\uparrow\rangle = |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} = e_1 \\ \\ \text{spin down } |\downarrow\rangle = |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} = e_2\end{array}\right\} \text{qubits as electron spin}$$

"Mystical" connection between $\mathbb{C}^2$ and $\mathbb{R}^3$

unit "sphere" $= \{u \in \mathbb{C}^2 : \|u\| = 1\}$

unit sphere in $\mathbb{R}^3$ $= \{v : \|v\| = 1\}$

Def: (1st def of a state): Let $\mathcal{H}$ be a $\mathbb{C}$-space (corresp. to a physical system). A state (pure state) of $\mathcal{H}$ is a unit vector in $\mathcal{H}$. ($u \in \mathcal{H}$ s.t. $\|u\| = 1$)

Ex: $\mathcal{H} = \mathbb{C}^2$: $u = \alpha|0\rangle + \beta|1\rangle$ such that

$$1 = \|u\| = \sqrt{\langle u, u\rangle} = \langle u, u\rangle = \big\langle \alpha|0\rangle + \beta|1\rangle, \alpha|0\rangle + \beta|1\rangle \big\rangle$$

$$= \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2$$

Def: Let $u, v \in \mathcal{H}$. Say $u \propto v$ if $u = \alpha v$

"u, v differ by an overall phase factor"

for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Equivalently
$\exists \theta \in \mathbb{R}$ s.t. $u = e^{i\theta} v$ (call $e^{i\theta}$ a global / overall phase factor)

Note: If $u \propto v$, then u & v represent the same physical state.

unit vectors $u \propto v$ iff u, v span the same 1-dim subspace of $\mathcal{H}$.

$$\underbrace{\{\alpha u : \alpha \in \mathbb{C}\}}_{\text{span of } u} = \underbrace{\{\beta v ; \beta \in \mathbb{C}\}}_{\text{Span of } v} \qquad [\beta := \alpha e^{i\theta}]$$

Upshot: pure states of $\mathcal{H}$ correspond one-to-one with 1-dim subspace of $\mathcal{H}$.

Def: Fix $\mathcal{H}$ arbitrary $\mathbb{C}$-space. A projector (orthogonal projector) on $\mathcal{H}$ is an operator $P \in \mathcal{L}(\mathcal{H})$ such that

1. $P = P^*$ (P is Hermitian)
2. $PP = P$ (P is idempotent)

Let $u \in \mathcal{H}$ be a unit (column) vector.

Then uu^* is a projector:

1. $(uu^*)^* = (u^*)^* u^* = uu^*$ (uu^* is Hermitian)

2. $(uu^*)(uu^*) = u(u^*u)u^* = \langle u,u\rangle uu^* = uu^*$ (3)
(b/c u is a unit vector).

Let P be a projector on $\mathcal{H}$.

Let $V := \mathrm{img}(P)$ $(:= \{Pu : u \in \mathcal{H}\})$

V is a vector subspace of $\mathcal{H}$ (closed under vector + and scalar $\times$)

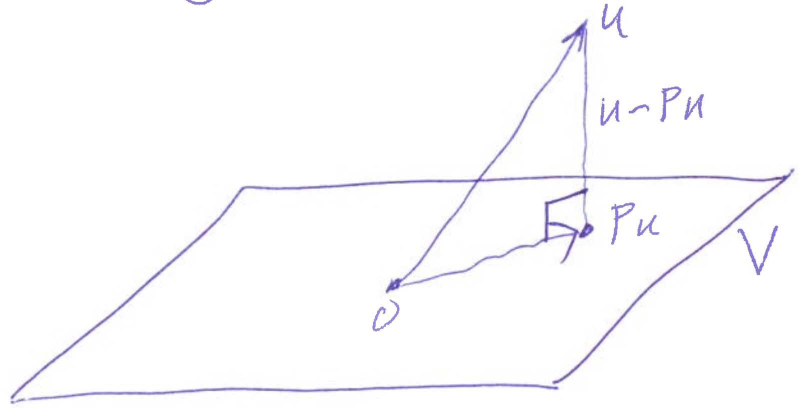
Fact: $\dim(V) = \mathrm{Tr}\, P$ (later)

Fact: P fixes V pointwise: $\forall u \in V$, $Pu = u$

Proof: $u \in V \Rightarrow \exists w \in \mathcal{H}$, $u = Pw$. So

$Pu = PPw = Pw = u$.

converse: For every subspace V of $\mathcal{H}$, there exists a unique projector P such that $V = \mathrm{img}(P)$.



P projector onto V

Show that $\langle u - Pu, Pu\rangle = 0$:

$$\langle u-Pu, Pu\rangle = \langle u, Pu\rangle - \langle Pu, Pu\rangle$$
$$= \langle u, Pu\rangle - \langle u, P^*Pu\rangle$$
$$= \langle u, Pu\rangle - \langle u, PPu\rangle$$
$$= \langle u, Pu\rangle - \langle u, Pu\rangle = 0 \quad \blacksquare$$

1-1 corresp between subspaces of $\mathcal{H}$ with their associated projectors:

$$\underset{\text{subspace}}{V \subseteq \mathcal{H}} \longleftrightarrow P \in \mathcal{L}(\mathcal{H})$$

s.t. $V = \text{img}(P)$.

$\therefore$ 1-1 corresp between physical states states and 1-d projectors (projectors with 1-dim images).

$\mathcal{H} = \mathbb{C}^2$ with $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} = e_1$, $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix} = e_2$

If V is 1-dim then corresponding projector is uu^*, for any unit vector $u \in V$:

$$(e^{i\theta}u)(e^{i\theta}u)^* = (e^{i\theta})(e^{i\theta})^* uu^* = |e^{i\theta}|^2 uu^* = uu^*$$

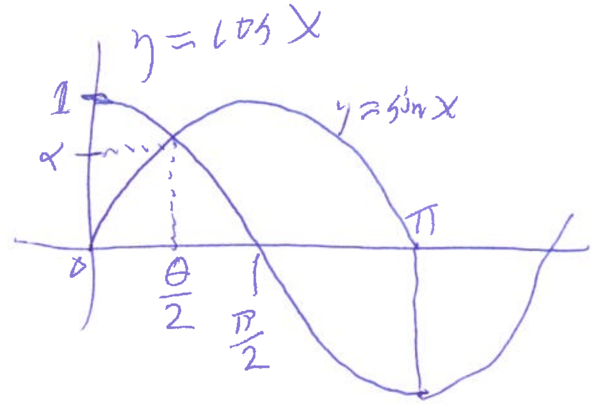
Better def of a pure state: a 1-dim projector on $\mathcal{H}$.

$\mathcal{H} := \mathbb{C}^2$. Arbitrary unit vector in $\mathbb{C}^2$ (up to a phase factor arbitrarily chosen):

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad (\alpha, \beta \in \mathbb{C}, \underbrace{|\alpha|^2 + |\beta|^2 = 1}_{\text{Know that } \alpha \le 1})$$

Choose phase factor so that $\alpha \geq 0$, so $0 \leq \alpha \leq 1$. So there exists a unique $\theta \in [0, \pi]$ such that

$$\alpha = \cos\left(\frac{\theta}{2}\right)$$



Then $|\beta|^2 = 1 - \alpha^2 = 1 - \cos^2\left(\frac{\theta}{2}\right) = \sin^2\left(\frac{\theta}{2}\right)$

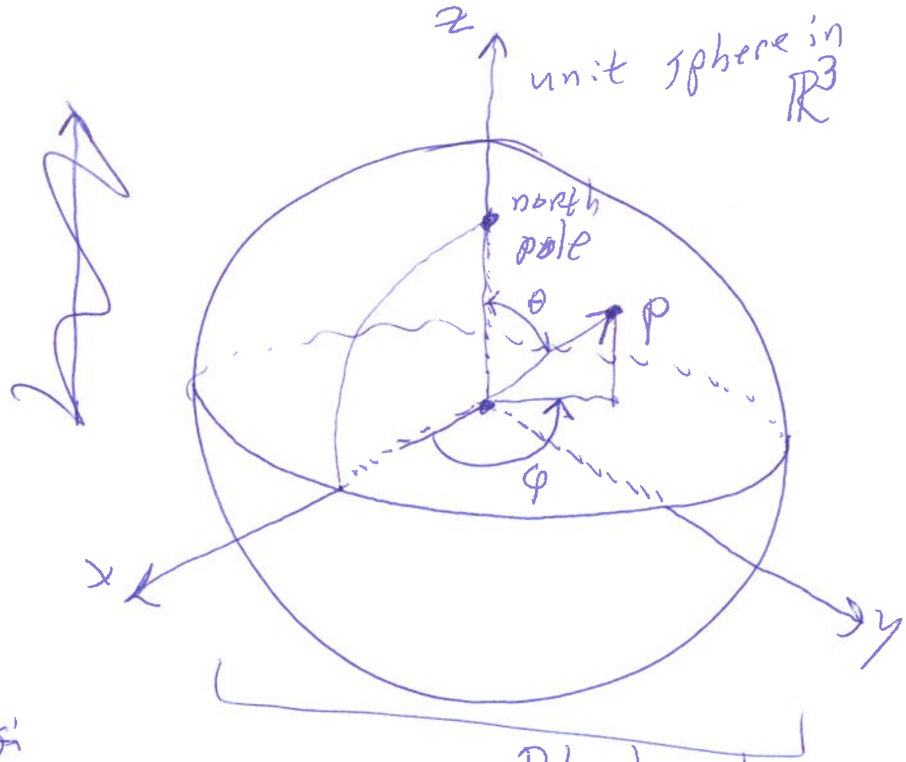
$\therefore$ $|\beta| = \sin\left(\frac{\theta}{2}\right)$

$\therefore$ $\beta = e^{i\varphi} \sin\left(\frac{\theta}{2}\right)$ some $\varphi \in [0, 2\pi)$

$\therefore$ $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi} \sin\left(\frac{\theta}{2}\right)|1\rangle$

θ, φ are the spherical coordinates of the spin direction.

$\theta =$ "colatitude" of p



Bloch sphere

$u = |\psi\rangle = \cancel{\cos\left(\frac{\theta}{2}\right) + e^{i\varphi} \sin}$

$$\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

is the spin state in direction given by θ, φ

$$u = |\psi\rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\text{pure state} = uu^* = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & e^{-i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$