CSCE 785
9/7/2023

Fix backlinks!

Today's agenda:

- Adjoints w/ examples
- Cyclic trace property
- Orthonormal bases
- $\mathcal{L}(\mathcal{H}, \mathcal{J})$

From last time: $A \in \mathbb{C}^{m \times n}$

Define adjoint A^* of A as the conjugate transpose of A: That is, for all $1 \le i \le m$ and $1 \le j \le n$,

$$[A^*]_{ji} = [A]_{ij}^*$$

Ex:

$$A = \begin{bmatrix} 2+3i & 4 & -2i \\ 1+i & 0 & 6 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 2-3i & 1-i \\ 4 & 0 \\ 2i & 6 \end{bmatrix}$$

$\forall\, u \in \mathbb{C}^n,\ v \in \mathbb{C}^m,\ \langle v, Au \rangle = \langle A^* v, u \rangle$

Basic properties:

$$\left.\begin{array}{ll} 1. & (A+B)^* = A^* + B^* \\ 2.\ a \in \mathbb{C}: & (aA)^* = \underline{a^*} A^* \end{array}\right\} \text{ * is conjugate linear}$$

3. $(AB)^* = B^* A^*$

4. $A^{**} = A$

so:

$$\left[\langle Av, u \rangle = \langle v, A^* u \rangle \right]$$

Trace property "cyclic": for any A, B s.t. AB ~~an~~ is square, then BA is square and

$$Tr(AB) = Tr(BA)$$

$$\therefore\ Tr(ABC) = Tr(CAB)$$

{note: Tr undefined for non-square matrices!}

$$\left.\begin{array}{l} Tr(A+B) = Tr\,A + Tr\,B \\ Tr(aA) = a\,Tr\,A \end{array}\right\} \text{ Tr is linear}$$

$$Tr\underbrace{[a]}_{\substack{1\times 1 \\ \text{matrix}}} = a$$

Orthonormal bases

Def: Let $\mathcal{H}$ be an n-dim $\mathbb{C}$-space

A set of vectors $\{b_1, \dots, b_k\} \subseteq \mathcal{H}$ is an <u>orthonormal set</u> if

$$\langle b_i, b_j \rangle = \delta_{ij}$$

[note $\Rightarrow k \leq n$ b.c. $b_1, \dots, b_k$ lin indep]

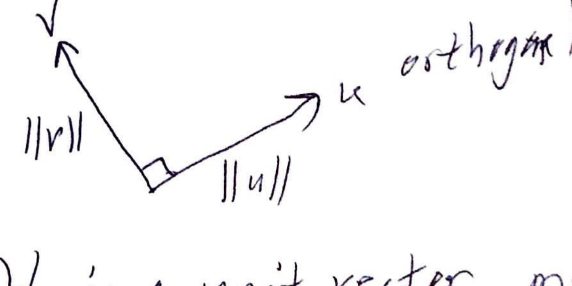
so $\|b_i\| = 1$ and b_i & b_j are orthogonal if $i \neq j$

$$\underbrace{\text{orthogonal}}_{\langle \cdot , \cdot \rangle = 0}$$

Note:

$\langle \cdot , \cdot \rangle$ restricted to real vectors (vectors in $\mathbb{R}^n$) is the same as the usual dot product

[adjoint = transpose for real vectors]



Def. $u \in \mathcal{H}$ is a <u>unit vector</u> means $\|u\| = 1$.

Def: An orthonormal basis $\{b_1, \dots, b_k\}$ is an orthonormal set that is also a basis. $[k = n]$

Standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ is orthonormal, but there are lots of other orthonormal bases for any $\mathbb{C}$-space.

<u>Thm</u>: (Gram-Schmidt) If $\{b_1, \dots, b_n\}$ is a basis for $\mathcal{H}$, then there exists a (unique) orthonormal basis $\{c_1, \dots, c_n\}$ such that

1. Each c_i is a lin combo (i.e. in the span of) of $\{b_1, \dots, b_i\}$, for $1 \le i \le n$

   [i.e. $\{c_1, \dots, c_i\}$ and $\{b_1, \dots, b_i\}$ span the same subspace]

2. $\langle b_i, c_i \rangle > 0$

won't really use ~~this~~ the conversion from b's to c's

$\mathbb{C}^{m \times n}$ is a $\mathbb{C}$-space: $A, B \in \mathbb{C}^{m \times n}$

then define $\langle A, B \rangle := \sum_{\substack{i,j \\ 1 \le i \le m \\ 1 \le j \le n}} [A]^*_{ij} [B]_{ij} = \operatorname{Tr}(A^* B)$

(✡)

Proof of (★):

$$\underbrace{\mathrm{Tr}(A^*B)}_{\substack{n\times n\\ \text{matrix}}} = \sum_{k=1}^{n} [A^*B]_{kk} = \sum_{k=1}^{n}\left(\sum_{j=1}^{m} [A^*]_{kj}[B]_{jk}\right)$$

$$= \sum_{j,k} \overline{[A]_{jk}}\,[B]_{jk}$$

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"Linear Maps = Matrices"

↑ ~~wrt~~ with respect to a choice of basis (bases)

Let $\mathcal{H}$, $\mathcal{J}$ be $\mathbb{C}$-spaces, n-dim & m-dim, respectively

Let $\{b_1, \dots, b_n\}$ be an orth basis for $\mathcal{H}$

Let $\{c_1, \dots, c_m\}$ " " " " " $\mathcal{J}$

A linear map $T: \mathcal{H} \to \mathcal{J}$ is uniquely represented by an $m\times n$ matrix over $\mathbb{C}$ as follows:

If $u = \sum_{j=1}^{n} u_j b_j \in \mathcal{H}$ $(u_j \in \mathbb{C})$

Then $Tu = \sum_{j=1}^{n} u_j (T b_j)$

(6)

$$Tb_j \in J$$

so $Tb_j = \sum_{k=1}^{m} w_{jk} c_k$ lin combo over $\{c_1, \dots, c_m\}$

Let M be the $m \times n$ matrix

such that $[M]_{kj} = w_{jk} \in \mathbb{C}$

Then letting $\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = M \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $Tu = \sum_{k=1}^{m} v_k c_k$.

So:

$$T \xleftrightarrow{\quad} M$$

choice of bases

$$Tb_j = \sum_{k=1}^{m} w_{jk} c_k \in J$$

$1 \leq i \leq m$:

$$\underline{\langle c_i, Tb_j \rangle} = \sum_{k=1}^{m} w_{jk} \underbrace{\langle c_i, c_k \rangle}_{\delta_{ik}}$$

$\langle \cdot, \cdot \rangle$ linear in 2nd arg

$$= \sum_{k=1}^{m} w_{jk} \delta_{ik} = w_{ji} = [M]_{ij}$$

Def: $\mathcal{H}, \mathcal{J}$ as above.

$\mathcal{L}(\mathcal{H}, \mathcal{J})$ is the vector space of all linear maps $\mathcal{H} \to \mathcal{J}$,

[$\cong$ all $m \times n$ matrices over $\mathbb{C}$, given choices of orthonormal bases for $\mathcal{H}$ and for $\mathcal{J}$]

$\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ (use same orthonormal basis for each) ~~ober~~ "linear operators ~~on~~ over $\mathcal{H}$"

a.k.a. "square operators" (rep. by square matrices)

Matrix defs lift to same def on linear maps via choice of orth. basis:

$W \in \mathcal{L}(\mathcal{H})$

$\operatorname{Tr} W = \operatorname{Tr} M$ [M represents W w.r.t. any orth basis]

indep of choice of basis.

Change of orthonormal basis

Let $\{b_1, \dots, b_n\}, \{c_1, \dots, c_n\}$ be o.n. bases for $\mathcal{H}$.

Let $U : \mathcal{H} \to \mathcal{H}$ be the unique linear map such that $Ub_j = c_j \quad \forall 1 \le j \le n$.

Then $\forall j, k$,

$$\delta_{jk} = \langle c_j, c_k \rangle = \langle Ub_j, Ub_k \rangle$$

$$= \langle b_j, U^*Ub_k \rangle = \delta_{jk} = \langle b_j, b_k \rangle$$

$$\therefore \langle b_j, U^*U b_k \rangle = \langle b_j, I b_k \rangle$$

$$\| \qquad\qquad \|$$

$$[U^*U]_{jk} \qquad [I]_{jk}$$

$$\therefore U^*U = I$$

<u>Def</u>: $U \in \mathcal{L}(\mathcal{H})$ is <u>unitary</u> if $U^*U = I$

(equivalently, $U^* = U^{-1}$)

(equivalently $UU^* = I$)

Fact $\langle Uv, Uw \rangle = \langle \underbrace{U^*U}_{I} v, w \rangle = \langle v, w \rangle$

$\forall v, w \in \mathcal{H}$

$\therefore$ Unitary operators are those that preserve the inner product.

Ex: $U = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$ unitary! (check)
