CSCE 785
8/31/2023

Probability (cont.)
Linear Algebra

HW1 due in a week (on dropbox)

Recall: A (discrete) probability space has a finite or countably infinite set Ω — elements called outcomes

sample space

probability measure

and a map $Pr : \overbrace{2^{\Omega}}^{\text{events}} \to \mathbb{R}$ s.t. some axioms

Ex: $\Omega = \{H, T\}$, $2^{\Omega} = \{\underset{0}{\emptyset}, \underset{1/2}{\{H\}}, \underset{1/2}{\{T\}}, \underset{1}{\{H,T\}}\}$

fair coin

A probability distribution is a list $(p_1, p_2, \ldots)$ of real numbers s.t.

$\forall i,\ p_i \geq 0$ and $\sum_i p_i = 1$

$\Omega = \{1, 2, \ldots\}$ setting $Pr\{E\} := \sum_{i \in E} p_i$ defines a probability measure

Prob dists $\longleftrightarrow$ Prob measures

Fix Ω, Pr. $E, F \subseteq \Omega$ are ~~independ~~

1. independent if $Pr[E \cap F] = Pr[E]Pr[F]$
2. mutually exclusive if $Pr[E \cap F] = 0$

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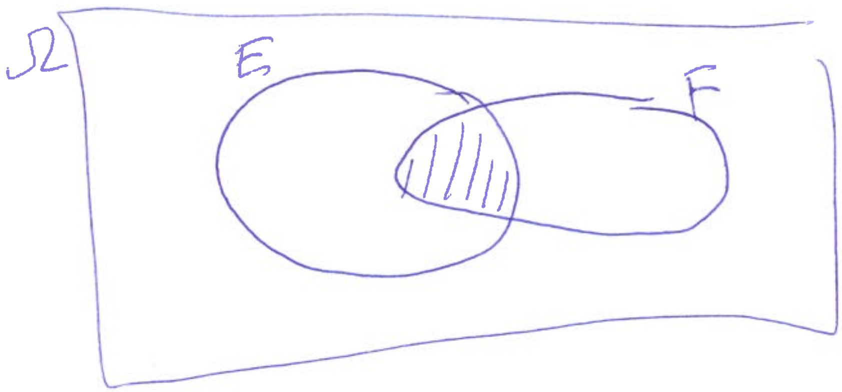
3. $Pr[E \mid F] := \dfrac{Pr[E \cap F]}{Pr[F]}$ (assume $Pr[F] > 0$)

conditional probability of E given F



Note E, F are indep. iff $P[E \mid F] = Pr[E]$

Random vars (real-valued) $\Omega = (\Omega, Pr)$ prob space.

A random variable (r.v.) on Ω is

a map $X : \Omega \to R$

Given $x \in R$, let $p_x := Pr\{\underbrace{\overbrace{X = x}^{\subseteq \Omega}}_{= \{a \in \Omega : X(a) = x\}}\} = Pr[x]$

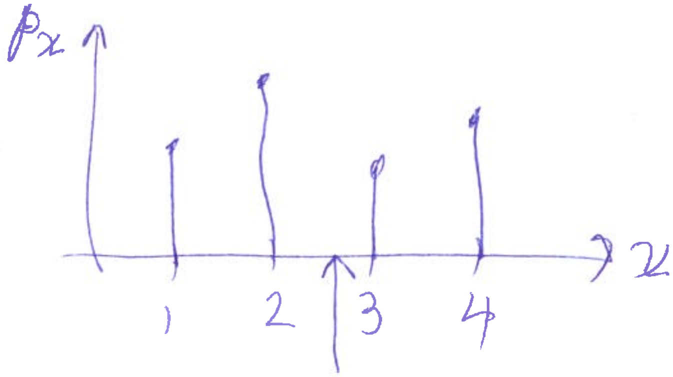
$\boxed{V} = \{X(a) : a \in \Omega\} = im(X)$

set of possible values $\{p_x\}_{x \in V}$ is a prob dist on V

Expected value: X is an r.v. with prob $\{p_x\}$

$$E(X) := \sum_x x\, p_x$$

weighted average of possible vals of X



Variance

$$\sigma_X^2 = V(X) := E\left((X-\mu)^2\right) = E(X^2) - \mu^2$$

where $\mu = E(X)$

$$\sigma_X := \sqrt{V(X)}$$

standard deviation

Linearity of expectation $a \in \mathbb{R}$, X, Y r.v.'s over Ω

$$E(X+Y) = E(X) + E(Y)$$

$$E(aX) = a\,E(X)$$

in general $E(XY) \neq E(X)E(Y)$

Independence of random vars. X, Y r.v.s over Ω

Say that X, Y are independent if $\forall x, y \in \mathbb{R}$, the events

$$\Pr\{a \in \Omega : X(a) = x \;\&\; Y(a) = y\} = \Pr[\{a : X(a) = x\}]\,\Pr[\{a : Y(a) = y\}]$$

$$\Pr\{X=x \ \& \ Y=y\} = \Pr\{X=x\}\cdot \Pr\{Y=y\}$$

Fact: If X, Y are indep, then $E(XY) = E(X)E(Y)$ and $V(X+Y) = V(X) + V(Y)$.

$E \subseteq \Omega$. Define r.v. $I\{E\}$ (I_E) such that

$\forall a \in \Omega$

$$I\{E\}(a) = \begin{cases} 1 & \text{if } a \in E \\ 0 & \text{if } a \notin E \end{cases}$$

Indicator r.v. of event E.

Fact: $E(I_A) = \Pr\{A\}$ $\quad (\forall A \subseteq \Omega)$

Linear Algebra — matrix algebra

Fix a field F (ex's: $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \{0,1\}$) $\quad$ $\mathbb{Z}_2 = \{0,1\}$

In $\mathbb{Z}_2$:

+	0	1
0	0	1
1	1	0

(same as XOR)

×	0	1
0	0	0
1	0	1

(same as logical AND)

$1+1=0$

$\Downarrow$

$1=-1$

$0=-0$

$+, -$ are the same operation over $\mathbb{Z}_2$

Fix $m, n \in \mathbb{N}$ $m, n > 0$.

Def: An $m \times n$ matrix (over F) is ~~an~~ a rectangular array of elements of F with m rows and n columns.

$\mathbb{Z}_2$: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ 2×3

Matrix ops: A, B both $m \times n$.

matrix addition — $A + B$ is $m \times n$ defined so that $\forall 1 \leq i \leq m$ $\forall 1 \leq j \leq n$

$$\underbrace{[A+B]_{ij}}_{(i,j)^{th} \text{ entry of } A+B} = [A]_{ij} + [B]_{ij}$$

Use "scalar" to mean any element of F

A & B are conformant (for matrix add) means $A + B$ is well-defined (true iff A, B have same dimension)

scalar multiplication: $a \in F$, A $m \times n$. Then aA s.t. $[aA]_{ij} = a[A]_{ij}$

Matrix multiplication

A $m \times n$

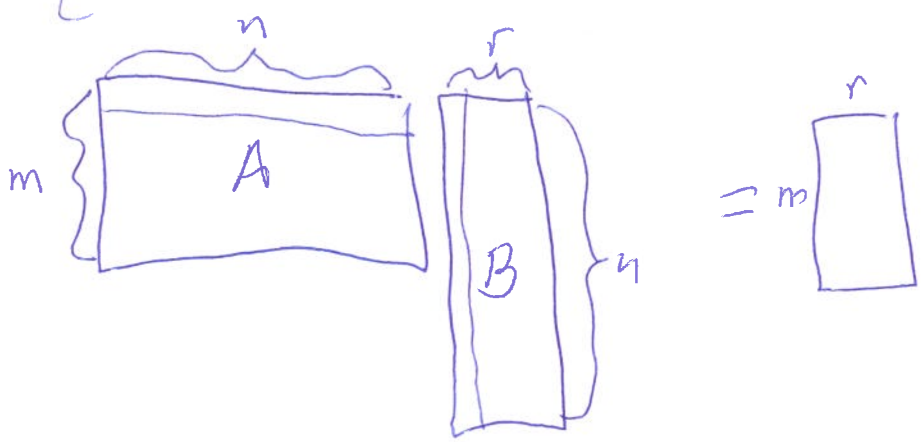
B $n \times r$ $(m, n, r > 0)$

Then AB ($m \times r$ matrix) is such that $\forall i, 1 \le i \le m$, $\forall k, 1 \le k \le r$

conformance

$$\underset{(m \times n)}{A}\ \underset{(n \times r)}{B} \qquad n = n \qquad m \times r$$

$$[AB]_{ik} := \sum_{j=1}^{n} [A]_{ij} [B]_{jk} .$$



A, B <u>commute</u> if $AB = BA$

Def: A <u>vector</u> is a matrix, one of whose dimensions is 1.

Column vector : $m \times 1$ matrix
(m-dim)

Row vector : $1 \times n$ matrix
(n-dim)

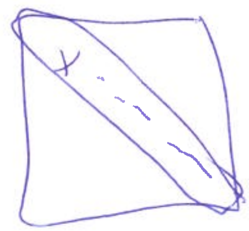
A 1×1 matrix $[a]$ identify with the scalar a

Square matrices

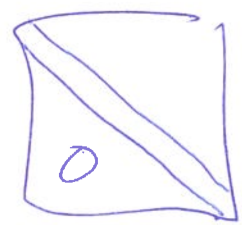
A matrix M is square if it is $n \times n$ for some n (n is the order of M)

Trace & Determinant

$$tr(A) = \sum_i [A]_{ii}$$



A square. A is upper triangular if $[A]_{ij} = 0$ for all $i > j$.



A is lower triangular ...... $i < j$

det maps $n \times n$ matrices to scalars such that

1. $\det(AB) = \det(A)\det(B)$
2. If A is triangular (upper or lower) then $\det(A) = \prod_i [A]_{ii}$

uniquely determines det