CSCE 785
8/29/2023

Today: Complex numbers
Elementary probability
~~Max~~ Matrix algebra

$i^2 = -1 \qquad i =$ imaginary unit

$\mathbb{C} = \{x + iy \;;\; x, y \in \mathbb{R}\}$

$z = x + iy \quad (x, y \in \mathbb{R})$

$x = \mathrm{Re}(z) \qquad\qquad 1 = 1 + 0i$ identity

$y = \mathrm{Im}(z)$

$z^* = \bar{z} = x - iy \qquad\qquad \mathrm{Re}(z) = \dfrac{z + z^*}{2} = x$

$\boxed{\dfrac{1}{i} = -i} \qquad\qquad \mathrm{Im}(z) = \dfrac{z - z^*}{2i} = y$

$\cancel{(-i)\left(\frac{1}{i}\right)} \qquad 1 = i\left(\frac{1}{i}\right) = i(-i) = 1$ ✓

$$zz^* = (x + iy)(x - iy) = x^2 - (iy)^2$$
$$= x^2 - i^2 y^2 = x^2 + y^2 \geq 0$$

"$z \geq 0$" implies $z \in \mathbb{R}$ and nonneg

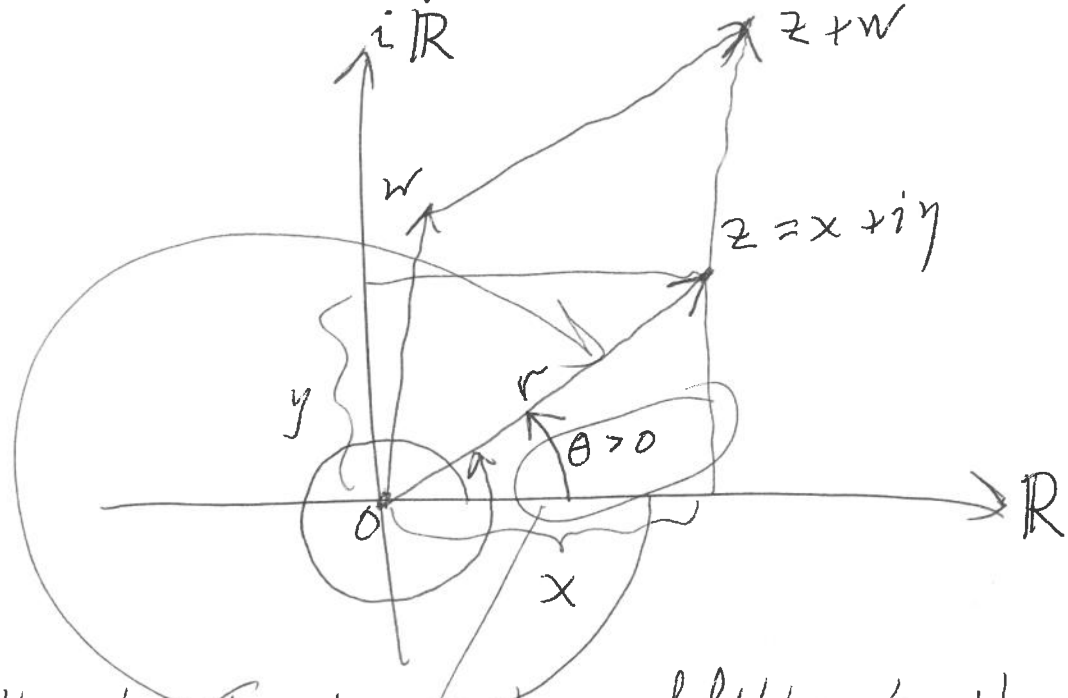
"$z > 0$" " " " positive

$(z \neq 0)$ $\frac{1}{z} = \frac{z^*}{x^2+y^2}$ $\quad$ $zz^* = x^2+y^2 = 0$ iff $z=0$

> 0 otherwise

Complex ~~pla~~ plane

Identify $z = x+iy$ with the point (x,y) in the cartesian plane



Addition in $\mathbb{C}$ is vector addition in the plane

Def: $|z|$ (norm of z or absolute value of z)

$$|z| := \sqrt{zz^*} = \sqrt{x^2+y^2} = \text{distance from } 0 \text{ to } z$$

Know $|z| \geq 0$ and equality holds iff $z=0$.

θ is called $\arg(z)$ (or $\arg z$)

θ is only determined up to additive multiples of 2π ③

$$\theta \equiv_{ct} \theta' \qquad (ct = \text{"coterminal"})$$

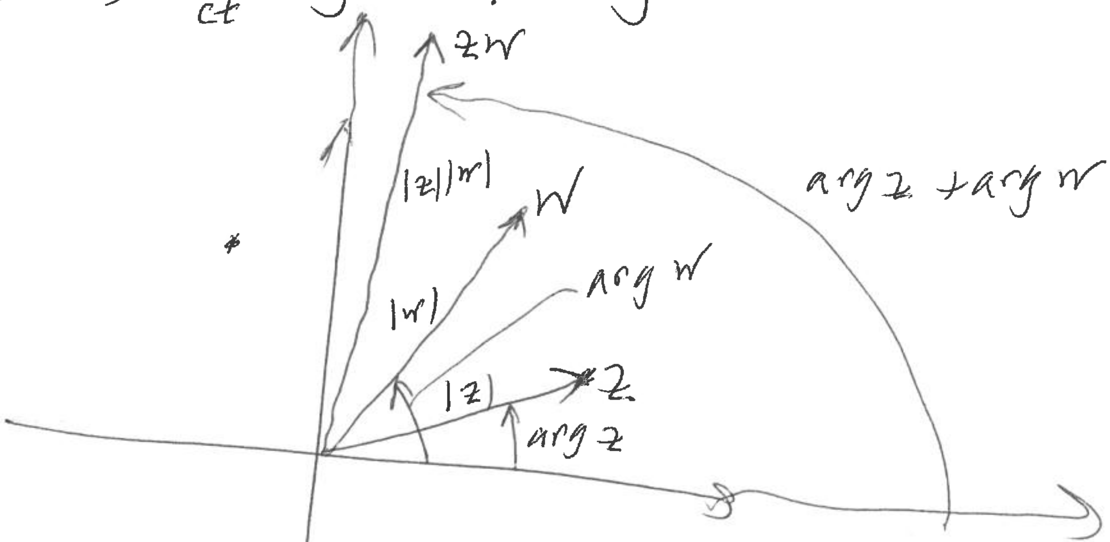
Pick a range for $\arg z$ to make it unique:

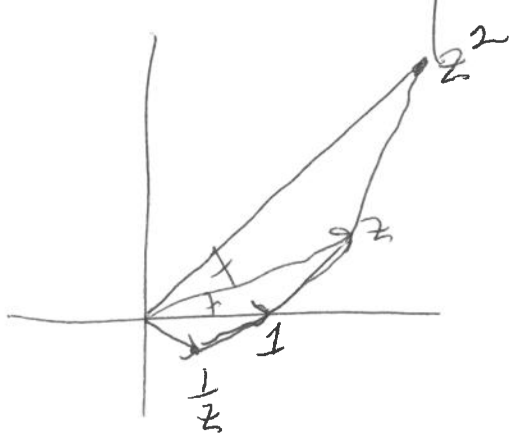
$$\arg z \in [0, 2\pi) \quad \text{(usually)}$$

$z, w \in \mathbb{C}$

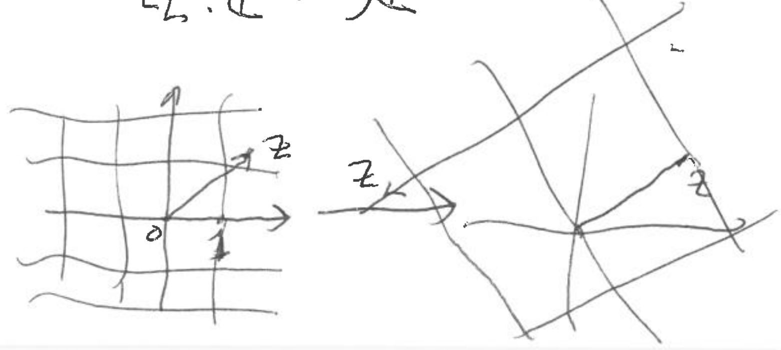
Fact: $|zw| = |z| \cdot |w|$

$$\arg(zw) \equiv_{ct} \arg z + \arg w$$





Fix $z \in \mathbb{C}$.
$z_L(w) = zw$
$z_L : \mathbb{C} \to \mathbb{C}$



Exponential map: $z \longmapsto e^z = \exp(z)$

$\mathbb{C} \to \mathbb{C}$

$z \in \mathbb{C}$: $$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

or $$e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n \quad (n \text{ pos int})$$

Fact: $e^0 = 1$, $e^{-z} = \frac{1}{e^z}$, $e^{z+w} = e^z e^w$

Euler's Formula: $\forall \theta \in \mathbb{R}$,

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

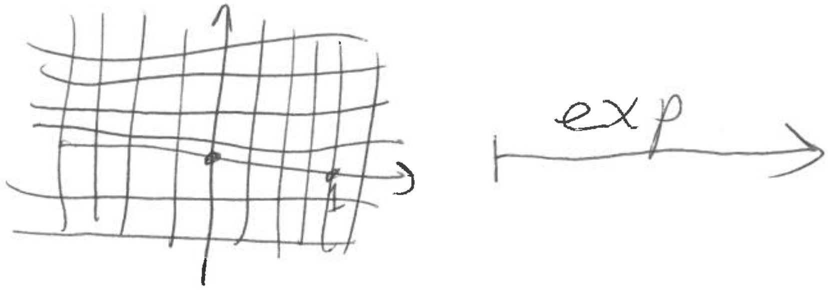
note: $$e^{-i\theta} = e^{i(-\theta)} \underset{\substack{\uparrow \\ \text{Euler}}}{=} \cos(-\theta) + i\sin(-\theta)$$

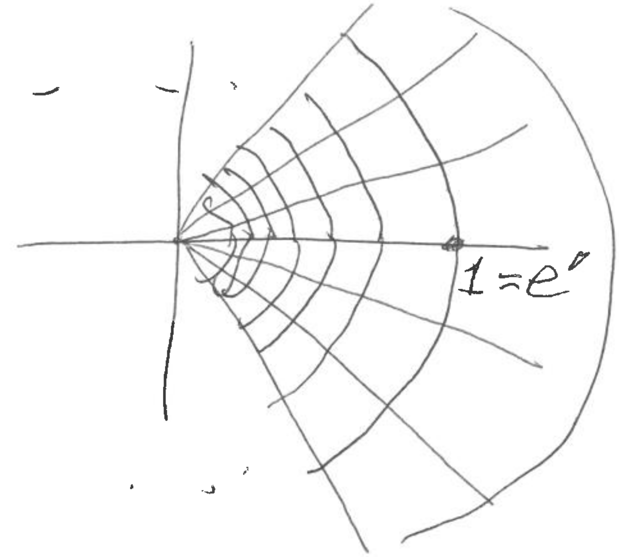
$$= \cos\theta - i\sin\theta$$

$$= \left(e^{i\theta}\right)^*$$

So

$$\boxed{\begin{aligned} \cos\theta &= \operatorname{Re}(e^{i\theta}) = \frac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta &= \operatorname{Im}(e^{i\theta}) = \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{aligned}}$$



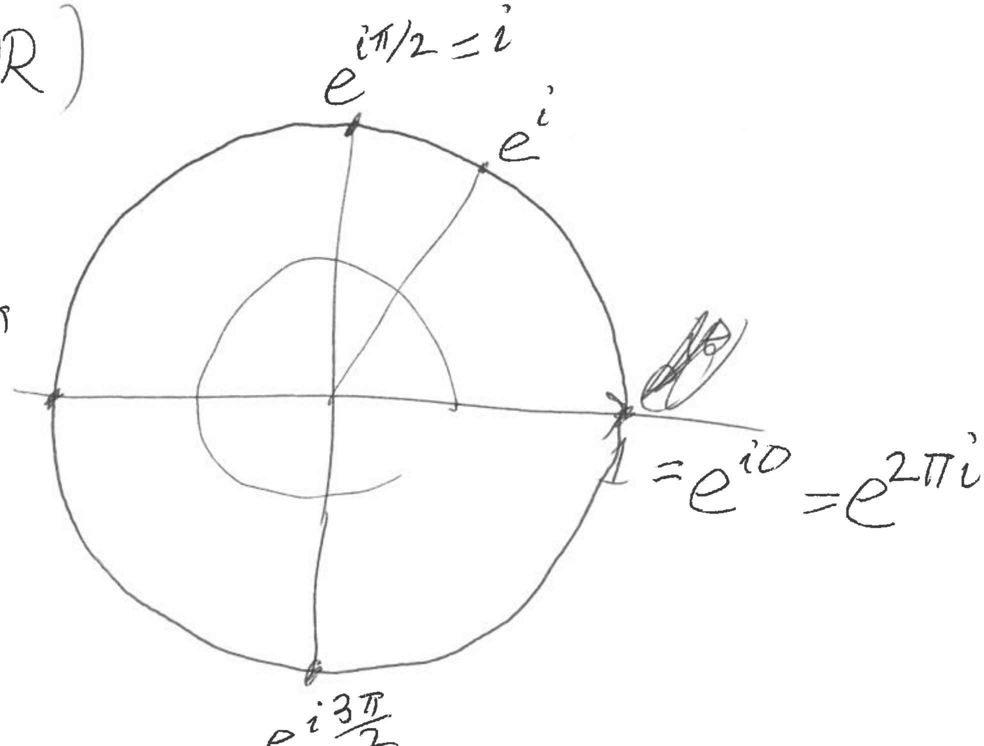


$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

$e^{i\theta}$. $(\theta \in \mathbb{R})$

$= \cos\theta + i\sin\theta$



$$e^{2\pi i} = e^0 = 1$$

More generally, $e^{z+2i\pi} = e^z e^{2\pi i} = e^z$

$\therefore$ exp has period $2\pi i$

$\text{range}(\exp) = \mathbb{C} \setminus \{0\}$

~~$z \neq 0$~~ let $z = x + iy$

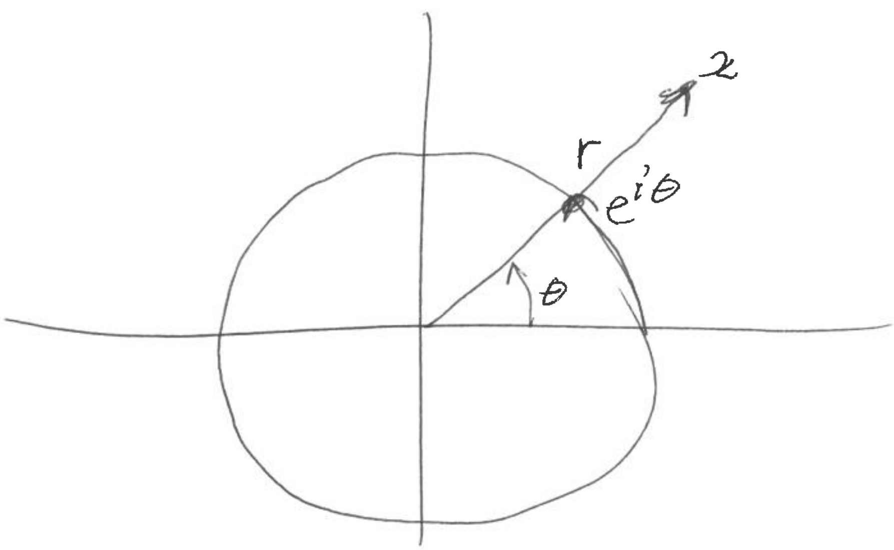
$$z = re^{i\theta} \qquad (r \geq 0,\ \theta \in \mathbb{R})$$

r, θ are polar coordinates of z:

$$r = |z| \qquad \left(|e^{i\theta}| = \cos^2\theta + \sin^2\theta = 1\right)$$

$$\theta \underset{\text{dt}}{=} \arg z$$

arg 0 is undefined



$\mathbb{C}$ is algebraically closed ("Fundamental Thm of Algebra"):

Every polynomial of positive degree with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n \qquad (n > 0)\ (c_n \neq 0)$$

$$\exists z_0 \in \mathbb{C} : p(z_0) = 0.$$

Corollary: any polynomial in $\mathbb{C}[z]$

coeffs in $\mathbb{C}$ — variable name

of degree n has n roots (not nec. distinct)

<u>Pf</u>: $p(z) = c_0 + \cdots + c_n z^n$

Pick z_0 s.t. $p(z_0) = 0$.

Then $p(z) = (z - z_0)\underbrace{q(z)}_{\deg n-1}$

Probability: A (discrete) probability space consists of a set Ω (finite or countable) an a function $\Pr : \underbrace{2^{\Omega}}_{\substack{\text{powerset} \\ \text{of } \Omega \\ = \{E : E \subseteq \Omega\}}} \to \mathbb{R}$ such that

1. $\Pr[E] \geq 0$ for all $E \subseteq \Omega$
2. $\Pr[\Omega] = 1$
3. If $A_1, A_2, \ldots \subseteq \Omega$ are pairwise disjoint, then

$$\Pr\left[\bigcup_{k=0}^{\infty} A_k\right] = \sum_{k=0}^{\infty} \Pr[A_k]$$ ("countable additivity")

Ω is the sample space

Pr is the probability function

$\Pr[E]$ = "the probability of E"
"the probability that E occurs"

Subsets of Ω are called events
Elements of Ω " " outcomes
(elementary events)

Ex: Coin flip $\Omega = \{H, T\}$

Fair coin: $\Pr[\{H\}] = \Pr[\{T\}] = \frac{1}{2}$

$\Pr[H] = \Pr[T] = \frac{1}{2}$

By additivity Pr is uniquely determined by its its values on individual outcomes:

$$\forall E \subseteq \Omega, \quad \Pr[E] = \sum_{x \in E} \Pr[x]$$

$$1 = \Pr[\Omega] = \sum_{x \in \Omega} \underbrace{\Pr[x]}_{p_x}$$

$\Omega = \{x_1, x_2, \ldots\}$

$p_{x_1} + p_{x_2} + \cdots + = 1$