Exps & logs

Sums

Asymptotic analysis

Def. Given any $a > 0$, $a \neq 1$, define $f(x) = a^x$

exponential function with base a.

Usually, $a > 1$

Usually, $a = 2$

$a = e \approx 2.718...$ "natural exp function"

Given any const $k \geq 0$

$$n^k \text{ is } O(2^n)$$

in fact, for constants $k < \ell$

$$n^k = o(n^\ell)$$

$$\therefore \quad n^k = o(2^n)$$

Def. f, g functions.

We say $f(n) = \mathrm{Poly}(g(n))$

means $\exists$ const k,

$$f(n) = O(g(n)^k)$$

Logarithms

Fix const $a > 0$, $a \neq 1$.

Define $\log_a x :=$ the unique β such that $a^\beta = x$.

$(x > 0)$.

(usually $a > 1$)

Properties: $a > 0$, $a \neq 1$, $x, y > 0$ reals, and $r \in \mathbb{R}$.

$$a^{x+y} = a^x a^y$$

$$a^{xy} = (a^x)^y$$

$$\left.\begin{aligned} \log_a(xy) &= \log_a x + \log_a y \\ \log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y \end{aligned}\right\} \text{product \& quotient rules}$$

$$\log_a 1 = 0$$

$$\log_a \frac{1}{x} = -\log_a x$$

$$\log_a x^r = r \log_a x \quad \text{(power rule)}$$

$b > 0$, $b \neq 1$.

$$\log_a b \cdot \log_b x = \log_a x$$

equiv:

$$\log_b x = \frac{\log_a x}{\log_a b} = \frac{\ln x}{\ln a}$$

$$\left[\begin{aligned} \ln x &= \log_e x \\ \lg x &= \log_2 x \end{aligned}\right]$$

So: If a, b constants > 1:

$$\log_b x = \Theta(\log_a x)$$

$$\log_a a^x = x \qquad (\forall x \in \mathbb{R})$$

$$a^{\log_a x} = x \qquad (\forall x > 0)$$

Follows that

$$\forall \varepsilon > 0, \log_a n = o(n^\varepsilon) \qquad \left(\forall \varepsilon > 0 \text{ constant}\right)$$

Ex: $n \lg n = o(n^{1.000001})$

Some sums

Arithmetic
Geometric } mixed arith-geom sum
Harmonic

Fix $k \geq 0$ $(k \in \mathbb{R})$

Arithmetic sum is the sum of the first k'th powers:

$$\sum_{j=1}^{n} j^k$$

$$k = 0: \quad \sum_{j=1}^{n} 1 = n$$

$$k = 1: \quad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$$k = 2: \quad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Thm: For any real $k \geq 0$,

$$A_k(n) := \sum_{j=1}^{n} j^k = \Theta(n^{k+1})$$

Proof: (1) $A_k(n) = O(n^{k+1})$ (upper bound):

$$\sum_{j=1}^{n} j^k \leq \sum_{j=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1})$$

(2) $A_k(n) = \Omega(n^{k+1})$

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$A_k(n)$

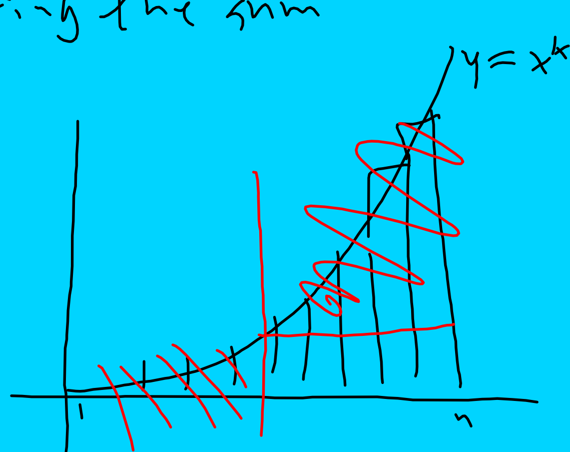
$$= \sum_{j=1}^{n} j^k \geq \sum_{j=\lceil \frac{n}{2} \rceil}^{n} j^k \geq \sum_{j=\lceil \frac{n}{2} \rceil}^{n} \left\lceil \frac{n}{2} \right\rceil^k$$

$$\geq \sum_{j=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k = \frac{1}{2^k} \sum_{j=\lceil \frac{n}{2} \rceil}^{n} n^k$$

$$\geq \frac{1}{2^k}\left(\frac{n}{2}\right) n^k = \frac{1}{2^{k+1}} n^{k+1}$$

$$= \Omega\left(n^{k+1}\right)$$

"splitting the sum"



Geometric sums: Fix $\underset{\text{constant}}{r \in \mathbb{R}}$, $r \neq 1$.

Finite geo sum:

$$\sum_{k=0}^{n-1} r^k = \boxed{\frac{r^n - 1}{r-1}} \underset{\text{if } r>1}{=} \Theta(r^n)$$

Infinite geo sum: $(|r| < 1)$

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} r^k$$

$$= \lim_{n\to\infty} \left(\frac{r^n - 1}{r-1}\right) = \boxed{\frac{1}{1-r}} = \Theta(1)$$

Harmonic sum:

$$H(n) = \sum_{j=1}^{n} \frac{1}{j} = \Theta(\lg n)$$

Proof: $\sum_{j=1}^{n} \frac{1}{j}$

$$= \underbrace{1}_{B_0} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{B_1} + \underbrace{\frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{j}}_{B_2} + \cdots + \frac{1}{n}$$

$$B_k = \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}-1}$$

$$H(n) \leq \sum_{k=0}^{\max} B_k$$

max $= B_k$ where $\underbrace{2^k \leq n}$ $< 2^{k+1}$

$k = \lg 2^k \leq \lg n$

so $H(n) \leq \sum_{k=0}^{\lfloor \lg n \rfloor} B_k \leq \sum_{k=0}^{\lfloor \lg n \rfloor} 1 \leq (\lg n) + 1$

$$= O(\lg n)$$

[For lower bound, $B_k \geq \frac{2^k}{2^{k+1}} = \frac{1}{2}$

Bound B_k from below by replacing each term in B_k with the 1st term beyond B_k.]

Integral approximation

Def. f is monotone ascending on an interval $[a, b]$ if

$$\forall x, y \in [a, b],\ x \leq y \Rightarrow f(x) \leq f(y)$$

f is strictly monotone increasing:

$$\forall x, y,\ x < y \Rightarrow f(x) < f(y).$$

f is monotone descending on $[a, b]$ if $-f$ is monotone ascending

$[x \leq y \Rightarrow f(x) \geq f(y)]$

strictly monotone decreasing . . .

$a > 1$ constant

a^x is strictly monotone increasing

$\log_a x$ - - - - - - - $(x > 0)$

x^k $(k$ const.$)$ is $\underset{x \geq 0}{}$ monotone ascending if $k \geq 0$

strict if $k > 0$.

Prop: Let $a \leq b$ be integers.

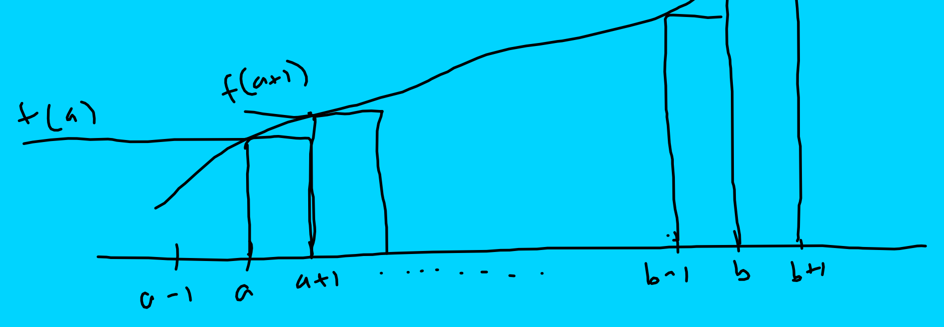
$f(x)$ monotone ascending on $[a-1, b+1]$ $(x \in \mathbb{R})$. Then

$$\int_{a-1}^{b} f(x)\,dx \leq \sum_{k=a}^{b} f(k) \leq \int_{a}^{b+1} f(x)\,dx.$$

$f(x)$ monotone descending on $[a-1, b+1]$, then

$$\int_{a}^{b+1} f(x)\,dx \leq \sum_{k=a}^{b} f(k) \leq \int_{a-1}^{b} f(x)\,dx$$

Picture proof: f ascending



$\sum_{k=a}^{b} f(k)$ = total area of the rectangles

$\leq$ total area under the curve from a to $b+1$

$$= \int_{a}^{b+1} f(x)\,dx$$

(for lower bound, shift rectangles to the left by 1.)

Mixed arith geo sum (infinite case)
($|r| < 1$)

$$S = \sum_{k=0}^{\infty} k r^k = \underbrace{\sum_{k=1}^{\infty} k r^k}$$

$$rS = \sum_{k=0}^{\infty} k r^{k+1} \underset{\ell = k+1}{=} \sum_{\ell=1}^{\infty} (\ell - 1) r^\ell$$

$$= \underbrace{\sum_{\ell=1}^{\infty} \ell r^\ell} - \sum_{\ell=1}^{\infty} r^\ell = S - r\left(\frac{1}{1-r}\right)$$

$$= \boxed{S - \frac{r}{1-r} = rS}$$

solve for S: $S(1-r) = \frac{r}{1-r}$

$$\boxed{S = \frac{r}{(1-r)^2}}$$