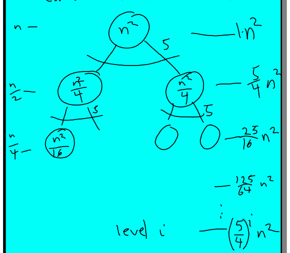
# Recurrences

Subst method example:

$$T(n) = 5T\left(\frac{n}{2}\right) + n^2$$

Recursion tree method:




$$T(n) = \sum_{i=0}^{\lg n} \left(\frac{5}{4}\right)^i n^2$$

$$= n^2 \sum_{i=0}^{\lg n} \left(\frac{5}{4}\right)^i = \Theta\left(n^2 \frac{\left(\frac{5}{4}\right)^{\lg n - 1} - 1}{\left(\frac{5}{4} - 1\right)}\right)$$

$$= \Theta\left(n^2 \left(\frac{5}{4}\right)^{\lg n}\right)$$

$$= \Theta\left(n^2 \cdot n^{\lg\left(\frac{5}{4}\right)}\right)$$

$$= \Theta\left(n^{2 + (\lg 5 - 2)}\right)$$

$$= \Theta\left(n^{\lg 5}\right)$$

Now consider

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2$$

. . . . .

$$T(n) = \sum_{i=0}^{\lg n} \left(\frac{3}{4}\right)^i n^2$$

$$\boxed{\begin{aligned} &= \Theta\left(n^2\left(\frac{1 - \left(\frac{3}{4}\right)^{\lg n}}{1 - 3/4}\right)\right) \\ &= \Theta\left(n^2 n^{\lg(3/4)}\right) \\ &\ \vdots \\ &= \Theta\left(n^{\lg 3}\right) \text{ ? no} \end{aligned}}$$

$$T(n) = \sum_{i=0}^{\lg n} \left(\frac{3}{4}\right)^i n^2$$

$$\le \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i n^2$$

$$= n^2 \underbrace{\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i}_{= \frac{1}{1 - \frac{3}{4}} = 4} = O(n^2)$$

$T(n) \ge n^2$ (obviously)

$\therefore T(n) = \Theta(n^2)$

Subst. method

Assume $n$ sufficiently large & $T(m) \le c \cdot m^2$ for all $\left(\frac{n}{2} \le m < n\right)$

[$c$ to be determined]

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2$$

$$\le 3c\left(\frac{n}{2}\right)^2 + n^2 \quad \text{(ind hyp.)}$$

$$= \frac{3}{4}cn^2 + n^2 = cn^2 - \frac{1}{4}cn^2 + n^2$$

$$\le cn^2$$

$T(n) = 5T(\frac{n}{2}) + n^2$

Subst method

Assume $T(m) \le cm^{\lg 5}$

$(m < n)$

$T(n) = 5T(\frac{n}{2}) + n^2$

$\le 5c(\frac{n}{2})^{\lg 5} + n^2$

$= \frac{5cn^{\lg 5}}{2^{\lg 5}} + n^2 = cn^{\lg 5} + n^2$

$\le cn^{\lg 5}$ Not true!

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Prove something stronger:

$T(n) \le cn^{\lg 5} - dn^2$

Assume $T(m) \le cm^{\lg 5} - dm^2$

$(m < n)$

$T(n) = 5T(\frac{n}{2}) + n^2$

$\le 5\left(c(\frac{n}{2})^{\lg 5} - d(\frac{n}{2})^2\right) + n^2$

$= cn^{\lg 5} - \frac{5}{4}dn^2 + n^2$

$\le cn^{\lg 5} - dn^2$ provided

$-\frac{5}{4}dn^2 + n^2 \le -dn^2$

$1 \le \frac{1}{4}d$

$4 \le d.$ ✓

Why no constraints on $c$?

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Master Theorem

(old weak version):

Suppose $T(n)$ is asymptotically positive, and for all sufficiently large $n$,

$$T(n) = aT(\frac{n}{b}) + f(n)$$

where $a > 0$, $b > 1$ are constants and $f(n)$ is asymptotically positive. Let $t = \log_b a$.

Then

Case1 (small $f$): If $f(n) = O(n^s)$ for some constant $s < t$, then

$T(n) = \Theta(n^t)$

Case2 (medium $f$): If $f(n) = \Theta(n^t)$ then

$T(n) = \Theta(n^t \lg n)$

[$t$ = "threshold"]

Case 3 (big $f$): If $f(n) = \Omega(n^u)$ for some constant $u > t$ and $\exists$ constant $\varepsilon > 0$ such that

$a f(\frac{n}{b}) \le (1-\varepsilon) f(n)$

for all sufficiently large $n$,

regularity condition

then

$T(n) = \Theta(f(n))$.

(Regularity condition can be assumed for all examples, e.g. $f(n) = n^u$ satisfies it $(u > t)$)

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Ex: $T(n) = 2T(\frac{n}{2}) + n$

$a = 2$

$b = 2$ $\quad t = \log_b a = \lg 2 = 1$

$f(n) = n = n^1 = n^t$

$\therefore$ case2: $T(n) = \Theta(n^t \lg n)$

$= \Theta(n \lg n)$

$T(n) = 3T\left(\frac{n}{2}\right) + n^2$

$a = 3$
$b = 2$ $\quad t = \log_2 3 = \lg 3$
$f(n) = n^2$ $\quad < \lg 4 = 2$

which case?
Case 3 (big f) $n = \Omega(n^2)$
$2 > \lg 3 = t$

$\therefore T(n) = \Theta(f(n)) = \Theta(n^2)$

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$T(n) = 5T\left(\frac{n}{2}\right) + n^2$

$a = 5$ $\quad t = \lg 5 > \lg 4 = 2$
$b = 2$
$f(n) = n^2$
$\therefore$ case 1 (small f):

$T(n) = \Theta(n^t) = \Theta(n^{\lg 5})$

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Ex:

$T(n) = 8T\left(\frac{n}{3}\right) + n^2$

$a = 8$ $\quad t = \log_3 8$
$b = 3$ $\quad < \log_3 9 = 2$
$f(n) = n^2$

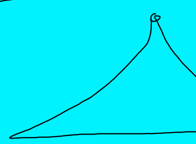
Note: $t = \log_b a$

$\iff b^t = a$.

Master thm proved by subst method.

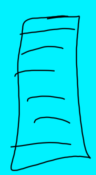
$T(n) = aT\left(\frac{n}{b}\right) + n^c$

case 1: tree method



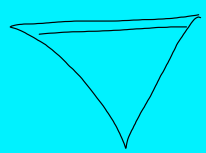
tree is leaf-heavy - most weight near the leaves

Case 2:



levels equal

case 3 : top-heavy tree



— $f(n)$

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Ex: $T(n) = 27T\left(\frac{n}{9}\right) + \underbrace{n\sqrt{n}}_{n^{1.5}}$

$b^a <,=,> c$? $\quad (f(n) = n^c)$

case 2: $T(n) = \Theta(n^{3/2} \lg n)$