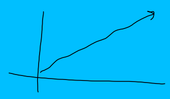
Let $f(n), g(n)$ be two functions (real-valued).

f is <u>monotone ascending</u> if $x \leq y \Longrightarrow f(x) \leq f(y)$



$f(x) = x^b$ ($b \geq 0$ constant) (domain $[0, \infty)$)

$f(x) = \log_b x$ ($b > 1$) $x > 0$

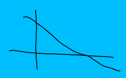
f is <u>strictly</u> <u>monotone</u> (increasing) if $x < y \Longrightarrow f(x) < f(y)$

<u>ex</u>: $f(x) = x^b$ ($b > 0$)

$f(x) = \log_b x$ ($b > 1$)

$f(x) = a^x$ ($a > 1$)

f is (strictly) <u>monotone</u> descending de<u>creasing</u> if $x \leq y \Longrightarrow f(x) \geq f(y)$



(strict) $x < y \Longrightarrow f(x) > f(y)$

<u>ex</u>: $f(x) = x^b$ $(b \underset{<}{\leq} 0)$ ($x > 0$)

<u>Fact</u>: If f is monotone ascending, then $-f$ is monotone descending. (and vice versa).

<u>Proof</u>: Assume f mono. asc. and $x \leq y$. Then $f(x) \leq f(y)$.

Then $-f(x) \geq -f(y)$

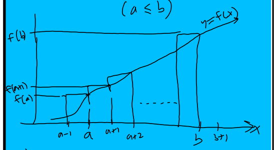
$\therefore -f$ is mono desc.

<u>Integral approximation to a sum.</u>

(1) Let f be mono. ascending.

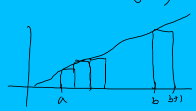
$$\int_{a-1}^{b} f(x)\,dx \leq \sum_{n=a}^{b} f(n) \leq \int_{a}^{b+1} f(x)\,dx$$

$(a \leq b)$



$\sum_{n=a}^{b} f(n)$ = area of the rectangles

$\int_{a-1}^{b} f(x)\,dx$ = area under the curve (a subregion of the rectangles)



Now rectangles are contained in the area under the curve.

How good is this?

$$\int_{a}^{b+1} f(x)\,dx - \int_{a-1}^{b} f(x)\,dx$$

$$= \int_{b}^{b+1} f(x)\,dx - \int_{a-1}^{a} f(x)\,dx$$

$$\leq f(b+1) - f(a-1)$$

Σ is confined to an interval of length $\leq f(b+1) - f(a-1)$

$$\sum_{j=1}^{n} \frac{1}{j} \quad (\underline{\text{Harmonic sum}})$$

$$\int \frac{dx}{x} = \ln x + C$$

$$\ln n = \int_1^n \frac{dx}{x}$$

$f(x) = \frac{1}{x} = x^{-1}$ mono. decreasing.

[If f is mono descending, then

$$\int_{a-1}^{b} f(x)\,dx \geq \sum_{n=a}^{b} f(n) \geq \int_a^{b+1} f(x)\,dx$$

upper bound on $\sum_{j=1}^{n} \frac{1}{j}$ is

$\int_0^n \frac{dx}{x}$ —— not good, it is $+\infty$.

Ideas?

$$\sum_{j=1}^{n} \frac{1}{j} = 1 + \sum_{j=2}^{n} \frac{1}{j}$$

$$\leq 1 + \int_1^n \frac{dx}{x} = 1 + \ln n$$

<u>Lower bound:</u>

$$\sum_{j=1}^{n} \frac{1}{j} \geq \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

So $\boxed{\ln(n+1) \leq \sum_{j=1}^{n} \leq 1 + \ln n}$

$$\sum_{j=1}^{n} = \Theta(\ln n) = \Theta(\lg n)$$

Asymptotic estimate of $\sum_{j=1}^{n} \frac{1}{j}$

$$\sum_{j=1}^{n} \frac{1}{j} = \overbrace{1}^{B_0} + \overbrace{\frac{1}{2} + \frac{1}{3}}^{B_1} + \overbrace{\frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{7}}^{B_2}$$

B_k has 2^k many terms:

$$B_k = \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}-1}$$

$$\leq \underbrace{\frac{1}{2^k} + \cdots + \frac{1}{2^k}}_{2^k \text{ terms}} = 1$$

$$\overbrace{1}+\overbrace{\frac{1}{2} + \cdots}\overbrace{\cdots}\overbrace{\cdots + \frac{1}{n}}$$

$$\sum_{j=1}^{n} \frac{1}{j} \leq \sum_{k=0}^{\lceil \lg n \rceil} B_k = \lceil \lg n \rceil + 1 = O(\lg n)$$

about $\lg n$ many blocks.

Lower bound:

$$\underbrace{1}_{C_0} + \underbrace{\frac{1}{2}}_{C_1} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{C_2} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{C_3} + \frac{1}{9} + \cdots$$

$$\sum_{j=1}^{n} \frac{1}{j} \geq \sum_{k=0}^{\lfloor \lg n \rfloor - 1} C_k \geq \sum_{k=0}^{\lfloor \lg n \rfloor - 1} \frac{1}{2} = \frac{1}{2} \lfloor \lg n \rfloor = \Omega(\lg n)$$

$$C_k = \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k} \quad 2^{k-1} \text{ terms}$$

$$\geq \frac{1}{2^k} \cdot 2^{k-1} = \frac{1}{2}$$

<u>Another sum:</u>

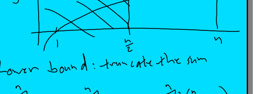
$$\sum_{k=1}^{n} \lg k \doteq ?$$

$$\lg k = \frac{\ln k}{\ln 2}$$

$$\sum_{k=1}^{n} \lg k = \frac{1}{\ln 2} \sum_{k=1}^{n} \ln k$$

[check $\int \ln x\,dx = x \ln x - x + C$

$$\sum_{k=1}^{n} \lg k \leq \underset{\#\text{terms}}{n}\ \underset{\text{biggest term}}{\lg n}$$



Lower bound: truncate the sum

$$\sum_{k=1}^{n} \lg k \geq \sum_{k=\frac{n}{2}+1}^{n} \lg k \geq \sum_{k=\frac{n}{2}+1}^{n} \lg\left(\frac{n}{2}+1\right)$$

$$\geq \sum_{k=\frac{n}{2}+1}^{n} \lg\left(\frac{n}{2}\right) = \frac{n}{2} \lg\left(\frac{n}{2}\right)$$

$$= \frac{1}{2} n (\lg n - 1) = \frac{1}{2} n \lg n - \frac{1}{2} n = \Omega(n \lg n)$$

$\therefore \sum = \Theta(n \lg n)$

$\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, b \geq 0$

show that

$$(n+a)^b = \Theta(n^b)$$

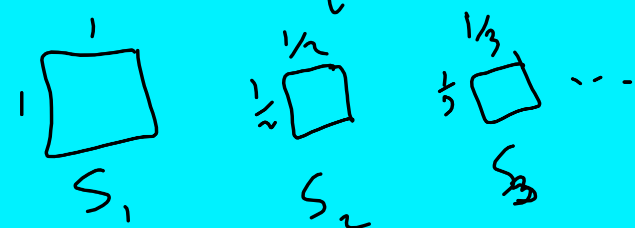
as $n \to \infty$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

<u>Open:</u>

Given squares



S_n is $\frac{1}{n} \times \frac{1}{n}$