Problem: Given a potentially biased coin:

$Pr[\text{heads}] := p \quad (0<p<1)$

$Pr[\text{tail}] := q = 1-p$

You may not know what p is.
Can flip as many times as you want.

You want to simulate a single **fair** (unbiased) coin flip.

repeat:
  flip twice
until outcome is different for the two flips

interpret the first of the pair as the unbiased flip.

Analyze the expected # of coinflips needed for this to work.

Def: A **Bernoulli trial** is a random experiment with two possible outcomes: "success" and "failure"

Let p be the prob of success
" q " failure

$p+q=1 \quad (q=1-p)$

Consider the following random variable:
run the same Bernoulli trial repeatedly (independently) until success.

G = # of trials for this to happen.

G is **geometrically distributed** with success prob p.
[must have $p>0$]

For any $k \geq 1$, find $Pr[G=k]$

$G=k$ means $k-1$ failures followed by success

$$\boxed{Pr[G=k] = \underbrace{q^{k-1}\, p}_{\text{by independence}}}$$

[Our example:
trial is 2 coin flips
success is either HT or TH
failure is HH or TT.]

$E(G)$ = expected value of G.

$$= \sum_{k=1}^{\infty} k \cdot Pr[G=k]$$

$$= \sum_{k=1}^{\infty} k q^{k-1} p$$

$$= p \underbrace{\sum_{k=1}^{\infty} k q^{k-1}}_{S}$$

$[\ell = k-1]$

$$S = \sum_{k=1}^{\infty} k q^{k-1} = \sum_{\ell=0}^{\infty} (\ell+1) q^{\ell}$$

$$qS = \sum_{k=1}^{\infty} k q^{k} = \sum_{k=0}^{\infty} k q^{k}$$

$$S - qS = \sum_{\ell=0}^{\infty} (\ell+1) q^{\ell} - \sum_{k=0}^{\infty} k q^{k}$$

$$= \underbrace{\sum_{\ell=0}^{\infty} \ell q^{\ell}} + \sum_{\ell=0}^{\infty} q^{\ell} - \underbrace{\sum_{k=0}^{\infty} k q^{k}}$$

$$= \sum_{\ell=0}^{\infty} q^{\ell} = \frac{1}{1-q} = \frac{1}{p}$$

$\therefore S - qS = \frac{1}{p}$

$$S = \frac{1}{p(1-q)} = \frac{1}{p^2}$$

So $E(G) = pS = \boxed{\frac{1}{p}}$

Our Example: $h = Pr[\text{heads}]$, $t = Pr[\text{tails}]$

$Pr[\text{success}] = Pr[HT \text{ or } TH]$
$= ht + th = 2ht$

$Pr[\text{failure}] = 1 - 2ht$

Expected number of repeat loop iterations is

$$\frac{1}{2ht}$$