Master Theorem (3rd Ed.)

Thm: Let $f(n)$ be asymptotically positive. Let $a>0$ and $b>1$ be real numbers. Define $T(n)$ by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Let $s := \log_b a$. Then

Case 1: ("Small $f(n)$"):
If $f(n) = O(n^t)$ for some contant $t<s$ then
$$\boxed{T(n) = \Theta(n^s).}$$

Case 2: ("Medium $f(n)$")
If $f(n) = \Theta(n^s)$, then
$$\boxed{T(n) = \Theta(n^s \lg n).}$$

Case 3: ("Large $f(n)$")
If $f(n) = \Omega(n^t)$ for some constant $t>s$ and $\exists \varepsilon > 0$, for all sufficiently large n,
$$af\left(\frac{n}{b}\right) \le (1-\varepsilon) f(n)$$
[regularity condition], then
$$\boxed{T(n) = \Theta(f(n)).}$$

Ex: Mergesort recurrence
$$T(n) = 2T\left(\frac{n}{2}\right) + n$$
$$\left[T(n) = aT\left(\frac{n}{b}\right) + f(n)\right] \leftarrow \text{M.T. hypothesis}$$
$$\left.\begin{aligned} a &= 2 \\ b &= 2 \end{aligned}\right] \begin{aligned} s &= \log_b a \\ &= \log_2 2 = 1 \end{aligned}$$
$f(n) = n = n^1 = n^s$

$\therefore$ Case 2:
$$T(n) = \Theta(n^s \lg n) = \Theta(n \lg n)$$

Ex: $T(n) = 8T\left(\frac{n}{3}\right) + n^2$
$$\left.\begin{aligned} a &= 8 \\ b &= 3 \end{aligned}\right] \begin{aligned} &s = \log_3 8 < \log_3 9 = t = 2 \\ &t = \log_3 9 = 2 \end{aligned}$$
$f(n) = n^2 = n^t$ $(t=2)$

$\therefore$ Case 3:
$$T(n) = \Theta(\underset{f(n)}{n^2})$$

Regularity condition?
$$af\left(\frac{n}{b}\right) \overset{?}{\le} (1-\varepsilon) f(n) \qquad [\text{const } \varepsilon > 0]$$
$$8\left(\frac{n}{3}\right)^2 \overset{?}{\le} (1-\varepsilon) n^2$$
$$\frac{8}{9} n^2 \overset{?}{\le} (1-\varepsilon) n^2 \quad \text{yes!} \; \left(\varepsilon := \frac{1}{9}\right)$$

Ex: $T(n) = 5T\left(\frac{n}{4}\right) + n$
$$\left.\begin{aligned} a &= 5 \\ b &= 4 \end{aligned}\right] s = \log_4 5 > \log_4 4 = 1 = t$$
$f(n) = n$, $t = 1$, $t < s$

$\therefore$ Case 1:
$$T(n) = \Theta(n^s) = \Theta(n^{\log_4 5})$$

$T_1(n) = 4T_1\left(\frac{n}{2}\right) + n$
$T_2(n) = 3T_2\left(\frac{n}{2}\right) + n$

T_1: $a=4$, $b=2$, $s = \log_2 4 = 2$
$f(n) = n = n^1$ $(t=1)$: $t<s$
$\therefore$ Case 1: $T_1(n) = \Theta(n^s) = \Theta(n^2)$

T_2: $a=3$, $b=2$, $s = \log_2 3$
$f(n) = n = n^1$ $(t=1)$ $t<s$
Case 1: $T_2(n) = \Theta(n^{\lg 3}) = o(n^2)$

Proof: $t < \log_b a = s$
$\iff b^t < b^{\log_b a} = a$
$t > \log_b a$
$\iff b^t > a$
$t = \log_b a \iff b^t = a$.

Case 3: $f(n) = \Omega(n^t)$ where $t > \log_b a$ $(b^t > a)$
& $af\left(\frac{n}{b}\right) \le (1-\varepsilon) f(n)$ $[\varepsilon > 0]$
WTS that $T(n) = \Theta(f(n))$
Obviously $T(n) = \Omega(f(n))$
$\left[T(n) = aT\left(\frac{n}{b}\right) + \underline{\underline{\underline{f(n)}}}\right]$
U.B.: Show that $T(n) = O(f(n))$
Substitution method

Inductive hypothesis:

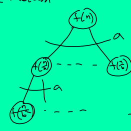
Assume that $T(m) \le cf(m)$ for all $m < n$.

$$T(n) = aT\left(\tfrac{n}{b}\right) + f(n)$$
$$\le acf\left(\tfrac{n}{b}\right) + f(n) \quad \text{[ind hyp.]}$$
$$= c\,\underbrace{af\left(\tfrac{n}{b}\right)} + f(n)$$
$$\le c(1-\varepsilon)f(n) + f(n)$$
$$= (c - \varepsilon c + 1)f(n) = cf(n) + (1-\varepsilon c)f(n)$$
$$\xrightarrow{\text{WTS}} \le cf(n)$$

provided

$$(1-\varepsilon c)f(n) \le 0$$
$$1 - \varepsilon c \le 0$$
$$1 \le \varepsilon c$$
$$\frac{1}{\varepsilon} \le c$$

Tree method



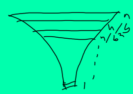
— $f(n)$

— $af\left(\frac{n}{b}\right) \le (1-\varepsilon)f(n)$

— $a^2 f\left(\frac{n}{b^2}\right) = a\underbrace{af\left(\frac{n}{b^2}\right)} \le a(1-\varepsilon)f\left(\frac{n}{b}\right) \le (1-\varepsilon)^2 f(n)$

$\vdots$

$$T(n) \le \sum_{i=0}^{\dots} (1-\varepsilon)^i f(n)$$
$$\le f(n) \underbrace{\sum_{i=0}^{\infty} (1-\varepsilon)^i}_{\text{finite constant}} = O(f(n))$$

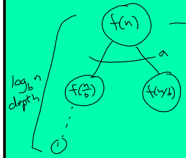


top-heavy tree

Case 2: $f(n) = \Theta(\underline{n^s})$ — assume n^s

$s = \log_b a$

$b^s = a$

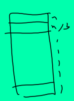


— $f(n) = n^s$

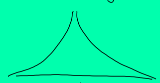
$= af\left(\frac{n}{b}\right) = a\left(\frac{n}{b}\right)^s = \frac{an^s}{b^s} = n^s$

$$T(n) = \underbrace{n^s}_{\text{common value for each level}} \cdot \underbrace{\log_b n}_{\#\text{levels}} = \Theta(n^s \lg n)$$

Balanced tree:



Case 1: Bottom-heavy tree:



Subst. method: $s := \log_b a$

$f(n) = O(n^t),\ t < s$

$b^t < a$

Attempt 1:

Assume $T(m) \le cm^s \quad (m < n)$

WTS $T(n) \le cn^s$

$$T(n) \le aT\left(\tfrac{n}{b}\right) + \underbrace{n^t}_{\text{assume } f(n) \le n^t}$$
$$\le ac\left(\tfrac{n}{b}\right)^s + n^t$$
$$= c\frac{a}{b^s}n^s + n^t \qquad \boxed{s = \log_b a,\ b^s = a}$$
$$= cn^s + n^t$$
$$\text{WTS} \le cn^s \quad \text{[won't work! for any } c > 0\text{.]}$$

Try again:

Assume $T(m) \le cm^s - dm^t$ (some constants c, d to be determined).

$$T(n) \le aT\left(\tfrac{n}{b}\right) + n^t$$
$$\le a\left(c\left(\tfrac{n}{b}\right)^s - d\left(\tfrac{n}{b}\right)^t\right) + n^t$$
$$= c\frac{a}{b^s}n^s - d\frac{a}{b^t}n^t + n^t$$
$$= cn^s - \left(d\left(\tfrac{a}{b^t}\right) - 1\right)n^t$$
$$\xrightarrow{\text{WTS}} \le cn^s - dn^t \quad \text{provided}$$

$\left(1 - d\left(\frac{a}{b^t}\right)\right)\cancel{n^t} \le -d\cancel{n^t}$

$d + 1 - d\left(\frac{a}{b^t}\right) \le 0$ # $1 \le d\underbrace{\left(\frac{a}{b^t} - 1\right)}_{>0}$

$\therefore d \ge \left(\frac{a}{b^t} - 1\right)^{-1} > 0$. #