Quiz 1

True or false:

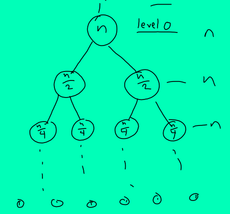
$$2^n = \Theta(2^{2n}).$$

Explain. FALSE

$$2^{2n} = 2^{n+n} = 2^n \cdot 2^n$$
$$> c \cdot 2^n$$

Recursion Trees

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

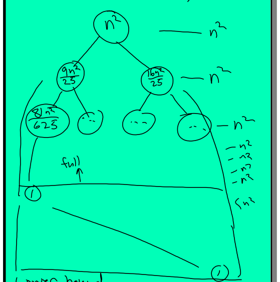


Total time = sum of the node values

In this case, each level has total value n, so

$$T(n) = n \cdot \underbrace{(\#\text{ levels})}_{\lg n} = n \lg n$$

$$T(n) = T\left(\frac{3n}{5}\right) + T\left(\frac{4n}{5}\right) + n^2$$



Lower bound

Full portion of the tree

$$= n^2 \underbrace{(\text{depth of the shallowest node})}_{\log_{\frac{5}{3}} n}$$

$$\leq T(n)$$

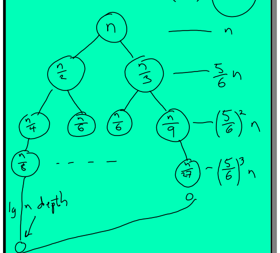
$$T(n) \geq n^2 \log_{\frac{5}{3}} n$$

Upper bound

$$T(n) \leq n^2 \cdot (\text{depth of tree})$$
$$= n^2 (\text{depth of deepest node})$$
$$= n^2 \log_{\frac{5}{4}} n = \Theta(n^2 \log_{\frac{5}{3}} n)$$
$$= \boxed{\Theta(n^2 \lg n)} = T(n)$$

Ex:

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{3}\right) + n$$



Upper bound

$$T(n) = \sum_{i=0}^{\lg n} n\left(\frac{5}{6}\right)^i \quad [\text{level } i]$$

$$= n \sum_{i=0}^{\lg n} \left(\frac{5}{6}\right)^i = \left(\frac{1 - \left(\frac{5}{6}\right)^{\lg n + 1}}{1 - \frac{5}{6}}\right) n$$

$$= 6n \underbrace{\left(1 - \left(\frac{5}{6}\right)^{\lg n + 1}\right)}_{< 1} \leq 6n$$

Lower bound

n (obvious from the recurrence & $T(k) \geq 0$)

$n = \Theta(6n)$, so

$T(n) = \Theta(n)$.

$T(n) = T(\frac{n}{2}) + T(\frac{n}{3}) + n$

Saw for the u.b.

$$T(n) \leq n \sum_{i=0}^{\lg n} \left(\frac{5}{6}\right)^i \qquad 0 \leq \frac{5}{6} < 1$$

$$\leq n \boxed{\sum_{i=0}^{\infty} \left(\frac{5}{6}\right)^i} = \Theta(n)$$

finite constant

$$= n\left(\frac{1}{1-\frac{5}{6}}\right) = 6n$$

<u>Change of variables</u>

$$T(n) = 2T(\underbrace{\sqrt{n}}_{n^{1/2}}) + \underline{\lg n}$$

Let $k := \lg n$, so $n = 2^k$

Let $S(k) = T(n) = T(2^k)$

$$S(k) = T(2^k) = 2T((2^k)^{1/2}) + k$$

$$= 2T(2^{k/2}) + k$$

$$= 2S(\tfrac{k}{2}) + k$$

$$S(k) = 2S(\tfrac{k}{2}) + k$$

So $S(k) = \Theta(k \lg k)$

$$T(n) = S(k) = S(\lg n)$$

$$= \Theta(\lg n \lg\lg n)$$

$T(n) = T(n-2) + n$

n — n

n−2 — n−2

n−4

n−6

⋮

0

$$T(n) = \sum_{i=0}^{n/2} (\underbrace{n-2i}_{2j})$$

$$\sum_{j=0}^{n/2} 2j = 2\left(\sum_{j=0}^{n/2} j\right) = 2\left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\right)$$

$$= \Theta(n^2)$$