Sums

Def: Let $a_1, a_2, \dots, a_n, \dots$ be numbers

$$\sum_{i=l}^{u} a_i = a_l + a_{l+1} + a_{l+2} + \cdots + a_u$$

If $u < l$, then this sum is 0. (empty sum).

$$\sum_{i=l}^{\infty} a_i := \lim_{n\to\infty} \sum_{i=l}^{n} a_i$$

Common sums:

$$\sum_{i=l}^{u} 1 = u - l + 1 \quad \left(\begin{matrix}\text{if}\\ u \geq l-1\end{matrix}\right)$$

Arithmetic sums

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \Theta(n^4)$$

generally, (k constant, $k \geq 0$),

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Proof: WTS $\underbrace{O(n^{k+1})}_{\text{upper bound}}$ and $\underbrace{\Omega(n^{k+1})}_{\text{lower bound}}$

U.B.: $\displaystyle\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k$ $\left(\begin{matrix} i^k \leq n^k \\ \text{if} \\ 1 \leq i \leq n\end{matrix}\right)$

$$= n^k \sum_{i=1}^{n} 1 = n^k \cdot n = n^{k+1}$$

$\therefore$ U.B. (constant can be 1)

L.B.

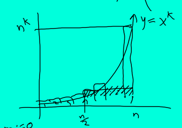
$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2}\rceil}^{n} i^k \quad \text{("truncating the sum")}$$

$$\geq \sum_{i=\lceil \frac{n}{2}\rceil}^{n} \left\lceil \frac{n}{2} \right\rceil^k \quad \text{(replace each term with the smallest term)}$$

$$\geq \sum_{i=\lceil \frac{n}{2}\rceil}^{n} \left(\frac{n}{2}\right)^k$$

$$\geq \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} n^{k+1}$$

$$= \Omega(n^{k+1}) \quad \left(\text{can choose } C := \frac{1}{2^{k+1}} > 0\right)$$

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sum := 0
for i := 1 to n
    for j := 1 to i
        sum++
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$\Theta(i)$

$$\sum_{i=1}^{n} \Theta(i) = \Theta\left(\sum_{i=1}^{n} i\right) = \Theta(n^2)$$

~~Geometric sums~~

~~Finite~~ geometric sum is of the form

$$\sum_{k=0}^{n-1} r^k = \boxed{\frac{r^n - 1}{r-1}}$$

$r \neq 1$ & is constant "ratio"

Proof: Let $S := \displaystyle\sum_{k=0}^{n-1} r^k$

$$= 1 + r + r^2 + \cdots + r^{n-1}$$

$$rS = r + r^2 + \cdots + r^{n-1} + r^n$$

$$rS - S = r^n - 1$$

$$(r-1)S$$

$$\therefore S := \frac{r^n - 1}{r-1} = \frac{1-r^n}{1-r}$$

Infinite geometric sum ($-1 < r < 1$) i.e., $|r| < 1$:

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} r^k$$

$$= \lim_{n\to\infty} \left(\frac{1 - r^n}{1-r}\right)$$

$$= \frac{1}{1-r} - \frac{1}{1-r} \underbrace{\lim_{n\to\infty} r^n}_{=0}$$

$$= \boxed{\frac{1}{1-r}}$$

for $(i=1; i<n; \overbrace{i *= 2}^{i=i*2})$

$\Theta(i)$ for $(j=1; j<=i; j++)$
sum++;

b'th iteration of the outer loop, $i = 2^{b-1}$

iterations $\approx$ k such that $2^{k-1} \doteq n$

solve for k:

$$\left(2^{k-1} = n\right) \text{ lg of both sides}$$

$$\underbrace{(k-1)\overbrace{\lg 2}^{1}}_{\text{power rule for logs}} = \lg n$$

$$k-1 = \lg n$$

$$k = \lg n + 1$$

Total time is $\displaystyle\sum_{b=1}^{\lg n+1} \Theta(\underbrace{2^{b-1}}_{\substack{i \\ \text{on b'th iteration}}})$

$$= \Theta\left(\sum_{b=1}^{\lg n+1} 2^{b-1}\right)$$

$$= \Theta\left(\sum_{m=0}^{\lg n} 2^{m}\right) \quad \boxed{m := b-1}$$

$$= \Theta\left(\frac{2^{\lg n+1}-1}{2-1}\right)$$

$$= \Theta\left(2^{\lg n+1}-1\right)$$

$$= \Theta\left(2^{\lg n+1}\right) = \Theta\left(2 \cdot 2^{\lg n}\right)$$

$$= \Theta(2n) = \Theta(n)$$

Harmonic sum

$$H(n) = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\lg n)$$

Proof: UB:

$$H(n) = \underbrace{\frac{1}{1}}_{B_0} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{B_1} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{B_2} + \underbrace{\frac{1}{8} + \cdots +}_{B_3}$$

$$B_i = \underbrace{\frac{1}{2^i} + \cdots + \frac{1}{2^{i+1}-1}}_{2^i \text{ many terms}} \leq \underset{\substack{\uparrow \\ \text{\# terms in } B_i}}{2^i} \cdot \underset{\substack{\uparrow \\ \text{largest term in } B_i}}{\left(\frac{1}{2^i}\right)}$$

$$= 1$$

$B_{\lceil \lg n \rceil}$ has $2^{\lceil \lg n \rceil} > 2^{\lg n} = n$ terms

so Blocks don't go beyond $B_{\lceil \lg n \rceil}$

$\therefore$ There are $\leq 1 + \lceil \lg n \rceil \leq \lg n$ many blocks

$$\text{So } H(n) \leq \sum_{i=0}^{\lceil \lg n \rceil} B_i \leq \sum_{i=0}^{\lceil \lg n \rceil} 1$$

$$= 1 + \lceil \lg n \rceil \leq 2 + \lg n$$

$$= O(\lg n)$$

L.B. similar block decomp:

$$B_i = \frac{1}{2^i} + \cdots + \frac{1}{2^{i+1}-1}$$

$$> 2^i \cdot \left(\frac{1}{2^{i+1}}\right) = \frac{1}{2}.$$

blocks: last $\overline{\text{full}}$ block starts with $\frac{1}{2^k}$ for largest k such that $\frac{1}{2^{k+1}-1} \geq \frac{1}{n}$

Solve for k:

$$2^{k+1} - 1 \leq n$$

$$2^{k+1} \leq n+1$$

$$k+1 \leq \lg(n+1)$$

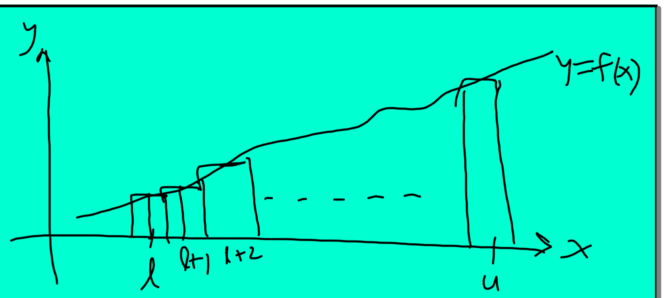
$$k \leq \lg(n+1) - 1$$

k is largest such, so

$$k = \lfloor \lg(n+1) - 1 \rfloor$$

$$\geq \lg n - 2$$

So



Then

$$\underbrace{\int_{l-1}^{u} f(x)\,dx}_{L.B} \leq \sum_{i=l}^{u} f(i) \leq \underbrace{\int_{l}^{u+1} f(x)\,dx}_{U.B.}$$

$$U.B. - L.B = \int_{u}^{u+1} f(x)dx - \int_{l-1}^{l} f(x)dx$$

$$\leq f(u+1) - f(l-1)$$

If f is monotone descending then

$$\int_{l}^{u+1} f(x)dx \leq \sum_{i=l}^{u} f(i) \leq \int_{l-1}^{u} f(x)dx$$

$$\underbrace{\int_{1}^{n+1} \frac{dx}{x}}_{} \leq \sum_{i=1}^{n} \frac{1}{i} = 1 + \underbrace{\sum_{i=2}^{n} \frac{1}{i}}_{}$$

$$\ln(n+1) = \Theta(\lg(n+1))$$

$$= \Theta(\lg n)$$

$$\leq 1 + \int_{1}^{n} \frac{dx}{x} = 1 + \ln n$$

$$= \Theta(\lg n)$$

Mixed arithmetic/geometric series)

$$\sum_{i=1}^{n} i^k r^i$$