CSCE 750

Asymptotic notation:

Recall — f, g functions that are positive-valued,

$f(n) = O(g(n))$ means "f grows no faster than g"

$\exists C, n_0, \forall n \geq n_0$

$$f(n) \leq C \cdot g(n)$$

$f(n) = \Omega(g(n))$ means "f grows at least as fast as g, asymptotically"

$\exists C > 0, \exists n_0, \forall n \geq n_0$

$$f(n) \geq C \cdot g(n)$$

$(\Leftrightarrow g(n) = O(f(n)))$

Think of $O$ as $\leq$

$\Omega$ as $\geq$

$f(n) = \Theta(g(n))$ means $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

"f and g are asymptotically similar"

Think of $\Theta$ as "="

Ex: $f(n) = 3n^2 + 7n - 6$

Claim: $f(n) = O(n^2)$

Pf: $3n^2 + 7n - 6$

$C := 4$ Then

WTS:

$$3n^2 + 7n - 6 \leq 4n^2$$
$$\Updownarrow$$
$$7n - 6 \leq n^2$$
$$\Uparrow$$
$$7n \leq n^2$$
$$\Updownarrow \ (n > 0)$$
$$7 \leq n$$

Set $n_0 := 7$. $\square$

Generally:

$$p(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_0 = O(n^d)$$

Claim:

$$3n^2 + 7n - 6 = \Omega(n^2)$$

Proof:

$$3n^2 + 7n - 6 \geq \underline{2n^2} \quad C := 2$$
$$\Updownarrow$$
$$n^2 + 7n - 6 \geq 0$$
$$\Updownarrow$$
$$n^2 + 7n \geq 6$$
$$\Uparrow$$
$$n^2 \geq 6 \quad (n > 0)$$
$$\Updownarrow$$
$$n \geq \sqrt{6}$$

set $n_0 := \sqrt{6}$ $\square$

Conclude: $3n^2 + 7n - 6 = \Theta(n^2)$

More notation

$O$ — "big Oh"

$\Omega$ — "big Omega"

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$o$ — "little oh"

$\omega$ — "little omega"

Think $o$ — $<$

$\omega$ — $>$

Def: $f(n), g(n)$ positive for all large enough $n$.

$f(n) = o(g(n))$

means $\forall C > 0, \exists n_0, \forall n \geq n_0$

$$f(n) \leq C \cdot g(n)$$

(implies $f(n) \neq \Omega(g(n))$ but $f(n) = O(g(n))$)

Equivalently, $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

$f(n) = \omega(g(n))$ means

$\forall C, \exists n_0, \forall n \geq n_0$

$f(n) \geq C \cdot g(n)$

equivalently, $g(n) = o(f(n))$

equivalently, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

$o$ — strictly slower growth ~~asympt~~ asymptotically

$\omega$ — " faster " "

Ex: $a, b \in \mathbb{R}$

$n^a = O(n^b) \iff a \leq b$

$n^a = o(n^b) \iff a < b$

$n^a = \Theta(n^b) \iff a = b$

$n^a = \Omega(n^b) \iff a \geq b$

$n^a = \omega(n^b) \iff a > b$

Ex code:

```
sum := 0
for i:=1 to n do
    for j:=1 to n do
        sum++;
    end-for
end-for
return sum;
```

$\Theta(1)$ $\Theta(n)$

do something taking $\Theta(n)$ time $n$ times,

$\Theta(n^2)$ time

```
sum := 0
for i:=1 to n do {
    for j:=1 to i do {
        sum++;
    }
}
return sum;
```

$\Theta(i)$

$\Theta(n^2)$ time!

consider

```
for i := n/2 to n do
    for j:=1 to n/2 do
        sum++
```

less work

counting the increments of sum

$\geq \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4} = \Omega(n^2)$

$\forall a > 1$, $f(n) := a^n$

exponential function with base $a$.

$f(n) = \omega(n^b)$ (any constant $b$)

$g(n) = e^n$ ($e \approx 2.71\ldots$)

natural exponential fcn

$e^n = \exp(n)$

~~logarithmic functions~~

$\forall a > 0, a \neq 1$

$\log_a n$ is the unique

$k \in \mathbb{R}$ such that

$a^k = n$

log & exp fcns are inverses of each other:

$\log_a a^x = x$ $(x \in \mathbb{R})$

$a^{\log_a y} = y$ $(y > 0)$

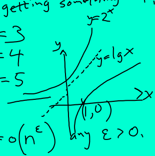
$\lg := \log_2$

$\lg n$ = # times can split $n$ in half before getting something $\leq 1$.

$\lg 8 = 3$

$\lg 16 = 4$

$\lg 32 = 5$




$\lg n = o(n^\varepsilon)$ any $\varepsilon > 0$.

$\forall a, b$, $a > 1$, $b > 1$

$\log_a n = \Theta(\log_b n)$

$\boxed{\log_a n = \frac{\log_b n}{\log_b a}}$ pos constant

Sum/product rules:

$$a^{x+y} = a^x a^y$$
$$a^{xy} = (a^x)^y$$
$$(ab)^x = a^x b^x$$
$$a^{-x} = \frac{1}{a^x} \qquad a^0 = 1$$

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$\log_a(xy) = \log_a x + \log_a y$

$\log_a\left(\frac{1}{x}\right) = -\log_a x$ product rule

$\log_a 1 = 0$

$\log_a x^r = r \log_a x$ power rule.

Prop: $a^{\log_b c} = c^{\log_b a}$

Proof: Take $\log_b$ of both sides:

$$\log_b\left(a^{\log_b c}\right) = \log_b\left(c^{\log_b a}\right)$$

$$\log_b c \log_b a = \log_b a \log_b c$$

Fact: If $\log_b x = \log_b y$ then $x = y$.

($\log_b$ is a one-to-one function.)

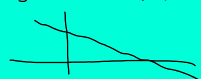
$\therefore$ Proposition. //

Def: $f : \mathbb{N} \to \mathbb{R}$ is monotone ascending if

$x \le y \Rightarrow f(x) \le f(y)$

monotone descending if

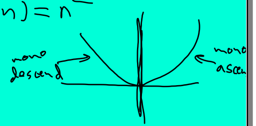
$x \le y \Rightarrow f(x) \ge f(y)$



$f$ is strictly monotone (ascending by default) means

$x < y \Rightarrow f(x) < f(y)$

$f(n) = n^2$




$\forall a > 1$, $f(x) = a^x$
$g(x) = \log_a x$

both strictly monotone (ascending).

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$\mathcal{O}(f)$    $f$ functio

$= \{g : g(n) = \mathcal{O}(f(n))\}$

$g(n) = O(f(n))$ means $g \in \mathcal{O}(f)$

If $S, T$ are sets of functio then

$S + T := \{f + g : f \in S, g \in$

$ST := \{fg : f \in S, g \in T\}$

$f + T := \{f\} + T$
$= \{f + g : g \in T\}$