csce750 — Analysis of Algorithms
Fall 2020 — Lecture Notes: Fibonacci Heaps

This document contains slides from the lecture, formatted to be suitable for printing or individual reading, and with some supplemental explanations added. It is intended as a supplement to, rather than a replacement for, the lectures themselves — you should not expect the notes to be self-contained or complete on their own.

1 Introduction

CLRS 19

A **Fibonacci heap** is a specific data structure that supports these operations:

- $\text{INSERT}(H, k)$
- $\text{MINIMUM}(H)$
- $\text{EXTRACTMINIMUM}(H)$
- $\text{UNION}(H_1, H_2)$
- $\text{DECREASEKEY}(H, x, k)$
- $\text{DELETE}(H, x)$

The primary advantages of a Fibonacci heap are the UNION and DECREASEKEY operations, which each take $\Theta(1)$ amortized time.

2 Binary heaps?

We *could* implement these operations using a binary heap (which we called a "heap" earlier this semester).

operation	**binary heap**	**Fibonacci heap**
	worst-case	amortized
$\text{INSERT}(H, k)$	$\Theta(\lg n)$	$\Theta(1)$
$\text{MINIMUM}(H)$	$\Theta(1)$	$\Theta(1)$
$\text{EXTRACTMINIMUM}(H)$	$\Theta(\lg n)$	$\Theta(\lg n)$
$\text{UNION}(H_1, H_2)$	$\Theta(n)$	$\Theta(1)$
$\text{DECREASEKEY}(H, x, k)$	$\Theta(\lg n)$	$\Theta(1)$
$\text{DELETE}(H, x)$	$\Theta(\lg n)$	$\Theta(\lg n)$

3 Fibonacci heap organization

A Fibonacci heap is a collection of min-heap ordered trees.

Each node x in each tree has these attributes:

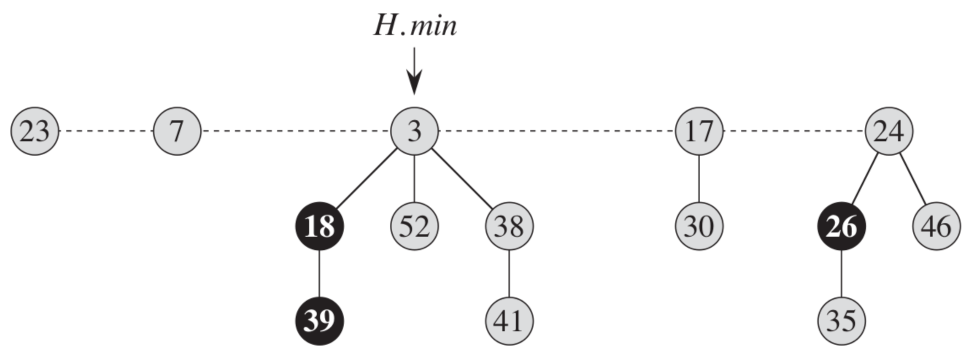
- x.key
- x.parent

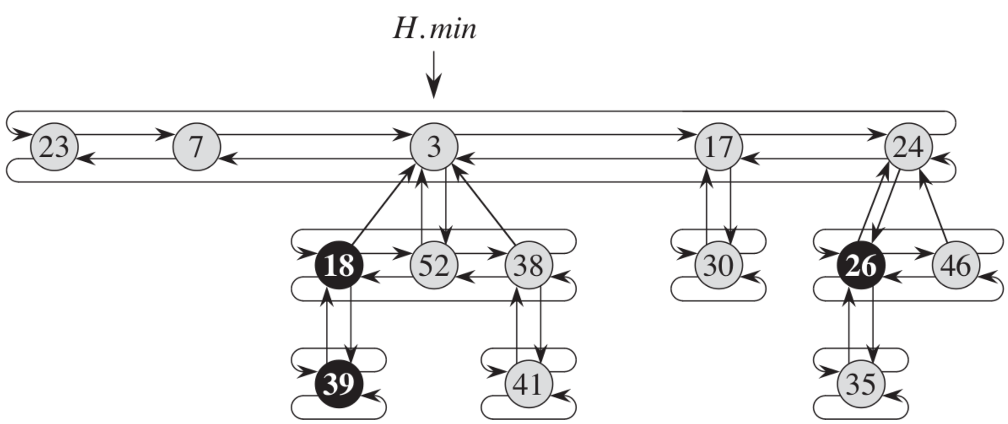
- x.child (a pointer to *any* of the children)
- x.left (a pointer to the left sibling)
- x.right (a pointer to the right sibling)
- x.degree
- x.mark[1]

The heap itself keeps this attribute:

- H.min – a pointer to the root of the tree containing the smallest key
- $H.n$ – the number of keys in the heap

4 Fibonacci heap example





5 Potential function

We'll analyze the Fibonacci heap data structure using the potential method for amortized analysis.

$$\Phi(H) = t(H) + 2m(H)$$

- $t(H)$ – number of trees in H

[1] Has x lost a child since the last time it was made the child of another node?

- $m(H)$ – number of marked nodes in H

In an application with multiple heaps that may be merged, use the total potential:

$$\Phi(H_{1..n}) = \sum_{i=1}^{n} \Phi(H_i)$$

Is this a valid potential function?

6 Fibonacci heap: Simple operations

INSERT(H, k):

- Create a new 1-node tree, and insert it as a sibling of H.min.
- Actual run time: $O(1)$
- Amortized run time: $\hat{c}_i = c + \Phi(H') - \Phi(H) = 1 + 1 = O(1)$

UNION(H_1, H_2):

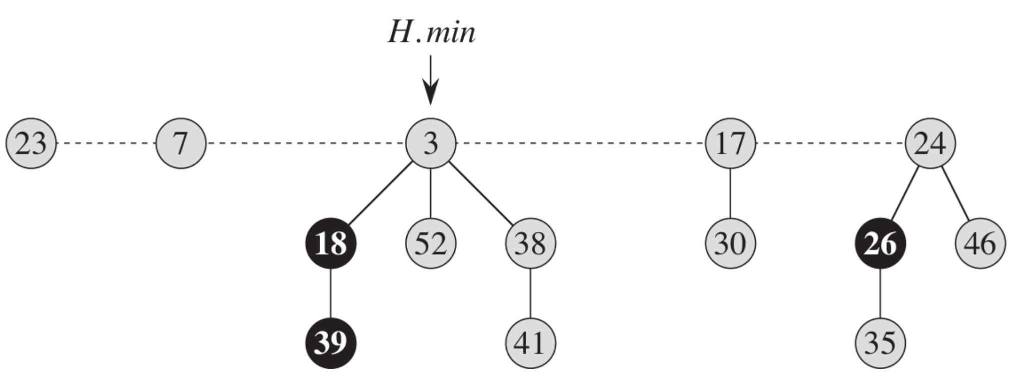
- Join the two linked lists of trees and select the new minimum.
- Actual run time: $O(1)$
- Amortized run time: $\hat{c}_i = c_i + \Phi(H') - (\Phi(H_1) + \Phi(H_2)) = 1 + 0 = O(1)$

7 Fibonacci heap: ExtractMin

EXTRACTMIN:

- Remove H.min from the list of trees.
- Promote each of the children of H.min to be top-level trees.
- "Consolidate" the heap, ensuring that no two trees have the same degree.
  - Use a direct address table, keyed on the degree of the root nodes.
  - Scan through the list of trees.
  - If we find two trees with the same degree d, **link** them, making one tree a child of the other, to create a combined tree with degree $d + 1$.

8 Consolidate



9 *ExtractMin analysis*

Let $D(n)$ denote the maximum degree of any node in a Fibonacci heap with n elements. (We'll show later that $D(n)$ is $O(\lg n)$.)

Amortized run time of EXTRACTMIN:

$$\begin{aligned}
\hat{c}_i &= \underbrace{D(n)+t(H)}_{\text{actual}} + \underbrace{\overbrace{D(n)+1}^{\text{number of trees}} + 2m(H)}_{\text{after}} - \underbrace{t(H) - 2m(H)}_{\text{before}} \\
&= O(D(n)) = O(\lg n)
\end{aligned}$$

10 *Fibonacci heap: DecreaseKey*

DECREASEKEY(x, k):

- If the new key is greater than the parent's key, update x.key and return.
- Otherwise, **cut** x from its parent, and add it as a new tree. Update H.min if needed.
- Use the mark attributes to promote any node that has lost two children since its last link to be a new tree. Search upward from former the parent of x toward the root for marked nodes. (**"cascading cut"**)

Idea: When a node loses its second child, promote it to the root level, to be folded into other trees on the next EXTRACTMIN.

Reminder: x.mark: Has x lost a child since the last time it was made the child of another node?

11 *Fibonacci heap: DecreaseKey analysis*

Let c denote the number of calls to CASCADINGCUT. Then $c-1$ trees were created by the cascading cuts.

- Actual cost: c
- Change in potential:
  - $t(H)$ increases by c.
  - $m(H)$ decreases by at least $c-2$.
  - $\Phi(H') - \Phi(H) = c - 2(c-2) = 4 - c$
- Amortized cost:

$$\hat{c}_i = c + 4 - c = O(1)$$

12 *Fibonacci heap: Delete*

DELETE(x):

- DECREASEKEY$(x, -\infty)$
- EXTRACTMIN(H)

Amortized Analysis: $O(1) + O(\log n) = O(\log n)$

13 *Bounding the maximal degree*

We still need to show that, in a Fibonacci heap with n nodes, the maximum degree of any node is $O(\log n)$.

Intuition: The only way to get a large degree is to have many descendants and the only way to get new descendants is during the consolidate step.

Lemma: Consider a node x with degree k. Let $y_1, \ldots, y_k$ denote the children of x in the order in which they where added. Then the degree of $y_i \geq i - 2$.

Proof: When y_i was linked to x, x already had $y_1, \ldots, y_{i-1}$ as children, so at the time, x.degree $\geq i-1$. The consolidate process only links nodes with equal degree, so we also have y_i.degree $\geq i-1$. Since then, y_i has lost at most one child, so now y_i.degree $\geq i - 2$.

14 *Fibonacci numbers*

Recall the **Fibonacci sequence**: $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$.

Let $\phi = \frac{1+\sqrt{5}}{2}$.

Lemma: $F_{k+2} = 1 + \sum_{i=0}^{k} F_i$ (Prove by induction on k.)

Lemma: $F_{k+2} \geq \phi^k$ (Prove by induction on k.)

15 *How small can a subtree in a Fibonacci heap be?*

Lemma: Let x be a node in a Fibonacci heap with degree k. Then

$$\text{size}(x) \geq F_{k+2}$$

Proof: Let s_i denote the smallest possible size for a node of degree i. Induction on k. Base cases, for $k = 0$ and $k = 1$, are trivial. Assume the result for $1, \ldots, k-1$ to show for k.

$$\begin{aligned}
s_k &\geq 2 + \sum_{i=2}^{k} s_{y_i.\text{degree}} \\
&\geq 2 + \sum_{i=2}^{k} s_{i-2} \\
&\geq 2 + \sum_{i=2}^{k} F_i = 1 + \sum_{i=0}^{k} F_i \\
&= F_{k+2}
\end{aligned}$$

16 *Maximum degree*

Finally, if k is the maximum degree in the heap, we have,

$$n \geq \text{size}(x) \geq F_{k+2} \geq \phi^k$$

which implies

$$\log_\phi n \geq k \qquad \Rightarrow \qquad k = O(\log n).$$