CSCE 551/MATH 562

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Outline

Practice with proofs

I. Automata & regular languages

II. Models of general computation (TM

III. Resource-bounded computatio

Polynomial time — P, NP

Polynomial space — PSPACE

Proof techniques:

Proof by ___
- induction
- cases
- construction

Lemma: (Pigeonhole Principle)

Let A, B be finite sets

$|A| = k$ (A is k-element

$|B| = \ell > k$

Let $f : B \to A$ be any funct with domain B & codomain A. f is not one-to-one, i.e.,

$(\exists x, y \in B)[x \neq y \;\&\; f(x) = f($

(Won't prove)

Def: Let G be a bipartite graph, where $G.V = L \dot\cup R$

L & R are nonempty set of vertices & $\dot\cup$ means disjoint union, that is, $G.V = L \cup R$ & $L \cap R = \emptyset$

$G.E \subseteq L \times R$ (every edge is of the form (u,v) for $u \in L$ & $v \in R$.

A complete matching $m : L \to R$ is such function, $(u, m(u)) \in G$

$\forall u \neq v \in L,\ m(u) \neq m(v)$

(m is 1-to-1)

Let $v \in L$ be a vertex.

The neighborhood of v i

$N(v) = \{w \in R : (v,w)$

For $S \subseteq L$, define

$$N(S) = \bigcup_{v \in S} N(v)$$

Theorem: Let G be a bipartite graph as ab Then a complete match exists for G if and iff

$\forall S \subseteq L$

Case 1: $\forall S \subseteq L$,

$|N(S)| > |S|$.

Then choose any $u \in L$, & choose any $v \in N(u)$.

We construct a complete matching $m : L \to R$ as follows:

define $m(u) := v$.

Let G' be the subgraph of G obtained by removing u & v from G (& any edges incident to u & v. G' is bipartite with $L' = L - \{u\}$

$R' = R - \{v\}$

For any $S \subseteq L'$, consider $N(S)$ in G'. Then

$N(S) = (N(S) \text{ in } G) - \{v\}$

so $|N(S) \text{ in } G'|$

$\geq |N(S) \text{ in } G| - 1$

$\geq |S|$ (assumed $|N(S) \text{ in } G| > |S|$

So by the inductive hyp, G' has a complete matching

$m' : L' \to R'$

Define $m(u') := m'(u')$ for all $u' \in L'$.

Then m is a complete matching of G. // Case 1

Case 2: Not case 1: $\exists$ nonempty proper $S \subseteq L$ such that

$|N(S)| = |S|$.

Both G_1 & G_2 have smaller lefthand side than G.

If G_1 & G_2 satisfy the hypothesis ($|N(S)| \geq |S|\ \forall S$), then done: get complete matchings m_1 for G_1 & m_2 for G_2 (by ind. hyp)

Remains to show that G_1 & G_2 both satisfy the hypothesis.

For G_1: Let $T \subseteq S$ be any subset of S.

Then $N(T)$ in G is a subset of $N(S)$ (in G).

so $N(T)$ in G_1 = $N(T)$ in G

so $|N(T) \text{ in } G_1| \geq |T|$

by the assumption in G. // G_1

$\therefore G_1$ satisfies the hyp.

For G_2, assume (for contradiction) that G_2 violates the hypothesis, that is, $\exists T \subseteq \overline{S}$ such that

$|N(T) \text{ in } G_2| < |T|$.

Consider $T \cup S$. Then

$(N(T \cup S) \text{ in } G)$

$= (N(T) \text{ in } G_2) \dot\cup (N(S) \text{ in } G)$

so

$|N(T \cup S) \text{ in } G| = |N(T) \text{ in } G_2 \cup N(S) \text{ in } G|$

$= |N(T) \text{ in } G_2| + \underbrace{|N(S)|}_{=|S|} < |T| + |S| = |T \cup S|$

So $|N(T \cup S) \text{ in } G|$

$< |T \cup S|$

contradicting our assumption about G.

$\therefore G_2$ must satisfy the hypothesis

$\blacksquare$

<u>Def</u>: An <u>alphabet</u> is any nonempty finite set.

If Σ is an alphabet, the elements of Σ are called <u>symbols</u> or <u>letters</u>

<u>Ex</u>: $\Sigma = \{0\}$ unary alphabet

$\Sigma = \{0, 1\}$ binary alphabet

etc.

$\Sigma = \{a, b, c\}$

A <u>string</u> over Σ is any finite sequence of symbols taken from Σ.

0111001 — binary string

000 — unary string

acbabbc —

ε — empty string