Recall: An editing system (over Σ) is a pair

$\langle \Sigma, \{(x_1,y_1), \ldots, (x_n,y_n)\} \rangle$

where Σ is an alphabet and each $x_i, y_i \in \Sigma^*$.

Given an editing system E (over Σ) and a string $w \in \Sigma^*$ an edit is a string w' that results from w by replacing some x_i substring with y_i. Say $w \Rightarrow w'$ (w edits to w')

Formally $w \Rightarrow w'$ if

$\exists i, 1 \le i \le n, \exists u, v \in \Sigma^*$ such that

$w = u x_i v$ and

$w' = u y_i v$

Editing problem (over Σ) is the language

$$EP_\Sigma := \{ \langle E, w \rangle : E \text{ is an editing system over } \Sigma, w \in \Sigma^*, \text{ and } \underbrace{w \Rightarrow \cdots \Rightarrow \varepsilon}_{\text{finite sequence of edits}} \}$$

Thm: There exists an alphabet Σ such that $A_{TM} \le_m EP_\Sigma$

[corollary: EP_Σ is undecidable]

Proof: Fix [universal TM]

$U := \langle Q, \tilde{\Sigma}, \Gamma, \delta, q_0, q_{acc}, q_{rej} \rangle$

such that $A_{TM} = L(U)$.

Let $\Sigma := Q \cup \Gamma \cup \{\$\}$

where \$ is symbol not in $Q \cup \Gamma$.

Fixed.

Given an instance $\langle M, w \rangle$ of A_{TM}, we construct (computably) an instance $\langle E, w' \rangle$ of EP_Σ such $\langle M, w \rangle \in A_{TM}$ iff $\langle E, w' \rangle \in EP_\Sigma$. As follows:

Given TM M and string w over M's input alphabet,

Let $v = \langle M, w \rangle$.

$v \in A_{TM} \iff U$ accepts v

$\iff M$ accepts w.

Construct $\langle E, w' \rangle$ as follows:

$w' = \$ q_0 v \$$

(initial ID of U on input v, bracketed with \$'s)

Add the following pairs (x, y) to E:

For every $q \in Q$ and $a \in \Gamma$ such that $\delta(q, a) = (r, b, R)$ add the pair (qa, br) to E.

For every $q \in Q$ and $a, c \in \Gamma$ such that $\delta(q, a) = (r, b, L)$, add the pairs (cqa, rcb) and $(\$qa, \$rb)$ to E.

(head in leftmost cell doesn't move for a L-transition).

For every nonhalting state $q \in Q$, add $(q\$, q\sqcup\$)$ to E.

For every $a \in \Gamma$, add

$(q_{acc}a, q_{acc})$ and

(aq_{acc}, q_{acc}) to E

Finally, add $(\$q_{acc}\$, \varepsilon)$ to E. End of construction.

[recall: $w' := \$q_0 v\$$]

To summarize, edits allow steps from ID's to successor ID's, and allow any accepting ID to edit to ε.

These are the only allowed edits, so $v \in A_{TM} \Leftrightarrow U$ accepts v

$\Leftrightarrow w' \Rightarrow \cdots \Rightarrow \varepsilon$. ▨

Other undecidable problems that don't mention machines:

<u>Post</u> <u>correspondence</u> <u>problem</u> (PCP):

Given a finite set of "dominoes"

$\left\{ \left[\frac{s_1}{t_1}\right], \left[\frac{s_2}{t_2}\right], \ldots, \left[\frac{s_n}{t_n}\right] \right\}$

where $s_1, \ldots, s_n, t_1, \ldots, t_n \in \Sigma^*$ for some alphabet Σ, is there a finite sequence of dominoes

$d_1 d_2 \cdots d_k$ drawn from the set s.t. the concat of the top strings equals the concat of the bottom strings?

Hilbert's 10th problem:

Given a polynomial $p(x_1, \ldots, x_n)$ with integer coefficients, is there a tuple $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $p(a_1, \ldots, a_n) = 0$?

Word problem for finitely generated groups.

Conway's Game of Life

⋮

Note PCP, EP_Σ are T-rec.

<u>Resource-bounded computability</u>

<u>Computational Complexity theory</u> studies the more fine-grained complexity of decidable problems. Attempts to classify such problems by their difficulty.

<u>Def</u>: Let M be a TM. Let $t: \mathbb{N} \to \mathbb{N}$ be a function. We say that <u>M runs in time t</u> [<u>or $t(n)$</u>] if, $\forall n \in \mathbb{N}, \forall w \in \underbrace{\Sigma^n}_{\text{string in } \Sigma^* \text{ of length } n}$,

M halts on input w in $\leq t(n)$ steps. [<u>worst-case time</u>].

M runs in time $O(t(n))$ if $\exists n_0, \exists C \geq 0, \forall n \geq n_0, \forall w, |w| = n$,

M on input w runs in $\leq Cn$ steps.

Our TM model is now multitape TMs for fine-grained complexity. We only work in a coarser-grained analysis where 1-tape and k-tapes are equivalent.

<u>Def</u>: M a TM. M runs in <u>polynomial time</u> if $\exists k$ constant such that M runs in time $O(n^k)$

Notation:

$f(n) \in \mathrm{Poly}(t(n))$ means $\exists k$

$f(n) \in O(t(n)^k)$.

Def: $\mathrm{DTIME}(t(n))$ is

the class of all languages decidable by TMs running in time $O(t(n))$.

Def: $$P := \bigcup_{k \in \mathbb{N}} \mathrm{DTIME}(n^k)$$

$$= \mathrm{DTIME}(\mathrm{Poly}(n)):$$

langs decided by TMs running in polynomial time.