1. (15 points) Show that if $A$ is regular, then the language

$$\text{PREFIX}(A) = \{w \in \Sigma^* | (\exists z \in \Sigma^*) wz \in A\}$$

is also regular. [Hint: If there is an $n$-state DFA recognizing $A$, then there also is an $n$ state DFA recognizing PREFIX($A$).]

2. (10 points) Show that every Turing-recognizable language is recognized by a Turing machine that never rejects any input, i.e., the machine always either accepts or loops. Your proof should include low-level detail about the machine’s states and transition function.

3. (15 points) Recall that the range of a function $f : A \rightarrow B$ is defined as

$$\{f(x) | x \in A\}.$$ 

Recall that every nonempty Turing-recognizable language is the range of some computable function. Using any method you like, precisely describe a computable function $f : \Sigma^* \rightarrow \Sigma^*$ whose range is exactly $A_{TM}$. A high-level description will suffice. [Hint: Fix a “fall-back” string $x \in A_{TM}$. Consider inputs to $f$ of the form $\langle M, w, t \rangle$ where $M$ is a TM, $w$ is a string over $M$’s input alphabet, and $t$ is a nonnegative integer. Don’t forget to say what $f$ does with inputs not of this form.]

4. (10 points) Show that the language

$$A = \{\langle M, w \rangle | M \text{ is a DFA, } w \text{ is a string, and } w^n \in L(M) \text{ for all } n \geq 0\}$$

is in P. (Here, “$w^n$” means $ww \cdots w$ ($n$ times).)
5. (15 points) Show that the language
\[ B = \{ \langle M, w \rangle \mid M \text{ is an NFA, } w \text{ is a string, and } w^n \in L(M) \text{ for all } n \geq 0 \} \]
is in PSPACE. [Hint: You may wish first to compute a number \( N \) such that \( \langle M, w \rangle \in B \) iff \( w^n \in L(M) \) for all \( 0 \leq n \leq N \).]

6. (15 points total) Let \( F \) be the quantified Boolean formula
\[ F = \exists x_1 \forall x_2 \exists x_3 \forall x_4 [(x_2 \lor x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_4}) \land (x_1 \lor x_3 \lor \overline{x_4})]. \]

(a) (10 points) Using the polynomial-time mapping reduction from TQBF to GG (Generalized Geography) given in the book or in class, give the instance of GG (i.e., the graph) corresponding to \( F \). You may either draw the graph, or else give a list of vertices and a list edges.

(b) (5 points) Is \( F \) true? Explain.

7. (15 points) Show that if \( \text{SAT} \in \text{NP} \), then \( \text{NP} = \text{coNP} \). (Recall that \( \text{coNP} = \{ \overline{A} \mid A \in \text{NP} \} \).) [Hint: To show this, the only thing you need to know about SAT is that it is \( \text{NP} \)-complete (under p-time m-reductions, of course).]

8. (15 points) Let \( \Sigma = \{0, 1\} \) and suppose that \( f : \Sigma^* \to \Sigma^* \) is some polynomial-time computable function. For each \( n \geq 0 \), let \( f_n \) be \( f \) restricted to the domain \( \Sigma^n \) of all binary strings of length \( n \). Show that the language \( L = \{ 0^n \mid n \geq 0 \text{ & range}(f_n) = \Sigma^n \} \) is in \( \text{coNP} \). [Hint: use the Pigeon Hole Principle.]