Answers to the CSCE 551 Final Exam, April 30, 2008

1. (15 points) Use the Pumping Lemma to show that the language

   $L = \{x \in \{0,1\}^* \mid \text{the number of 0s and 1s in } x \text{ differ (in either direction) by at most 2008}\}$

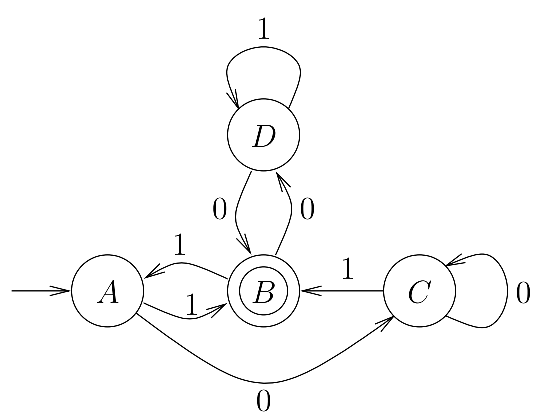
   is not regular.

   Answer: Given $p > 0$, let $s = 0^p1^{p+2008}$. Clearly, $|s| = 2p + 2008 \geq p$ and $s \in L$. If x, y, z are such that $xyz = s$, $|xy| \leq p$, and $|y| > 0$, then we must have $y = 0^m$ for some $m > 0$. Letting $i = 0$, we remove m zeros to get

   $$xy^iz = xy^0z = xz = 0^{p-m}1^{p+2008} \notin L.$$

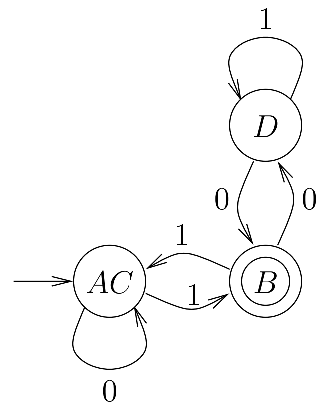
   Thus by the Pumping Lemma, L is not regular. (There are many other workable choices of s and i.)

2. (10 points) The following DFA is not minimum:



   Using any method you like, find the minimum equivalent DFA. (Draw its transition diagram.)

   Answer: States A and C are the only pair of indistinguishable states. Combining them, we get the minimum DFA:



3. (15 points) Give an m-reduction from A_{TM} to the language

$$B = \{\langle R\rangle \mid R \text{ is a TM that accepts a string } w \text{ iff } |w| = 2008\}.$$

Answer: Let

f = "On input $\langle M, w\rangle$ where M is a TM and w is a string:

(a) Let R = 'On input x:
   i. If $|x| \neq 2008$ then reject
   ii. Simulate M on w (and do what M does)'

(b) Output $\langle R\rangle$"

Then f m-reduces A_{TM} to B.

4. (25 points total) Fix some enumerator E. Let A be the language of all strings w such that

- E prints w at least once, and
- w is at least as long as any string that E prints before it first prints w.

Show:

(a) (15 points) A is decidable.

(b) (10 points) If E enumerates an infinite language, then A is infinite.

[This proves that every infinite Turing recognizable set has an infinite decidable subset.]

Answer: Let L be the language enumerated by E. It follows from the definition that $A \subseteq L$.

(a) There are two cases:

L **is finite:** Then A is also finite. Since every finite language is decidable (regular even), then A is decidable.

L **is infinite:** Let D implement the following algorithm:

"On input w:

i. Run E until it either prints w or it prints some string longer than w, whichever comes first

ii. If E prints w, then accept

iii. If E prints a string longer than w, then reject"

Since E enumerates an infinite language (by assumption), it must print strings that are arbitrarily long. Thus on any input w, Step (i) will eventually complete, and so D is a decider. It is also straightforward to see that $w \in A$ iff D accepts w. Thus D decides A, and so A is decidable.

[Note that in general we cannot compute, given $\langle E \rangle$, which case holds—finite or infinite. This question is undecidable, but it does not affect the decidability of A.]

(b) Assume that L is infinite. For any length $n \geq 0$, there are only finitely many strings of length $\leq n$, and so E must eventually print a string longer than n. The *first* such string printed by E is in A (clearly). So A contains a string longer than n. Since n is arbitrary, A must contain arbitrarily long strings, so A must be infinite.

5. (10 points) Prove that if TQBF $\in$ NP, then NP = PSPACE.

Answer: We already know that NP $\subseteq$ PSPACE (this was proven in class), so it suffices to show that PSPACE $\subseteq$ NP. Let A be any language in PSPACE. We know that TQBF is PSPACE-complete, so in particular, TQBF is PSPACE-hard, which means $A \leq_{\mathrm{m}}^{\mathrm{p}}$ TQBF because $A \in$ PSPACE. Now suppose that TQBF $\in$ NP. We proved in class that for any languages C and D, if $C \leq_{\mathrm{m}}^{\mathrm{p}} D$ and $D \in$ NP, then $C \in$ NP. In this case, we have $A \leq_{\mathrm{m}}^{\mathrm{p}}$ TQBF and TQBF $\in$ NP (by assumption), and thus $A \in$ NP. Since A was any PSPACE language, this shows that PSPACE $\subseteq$ NP.

6. (15 points) Let

$$L = \{\langle M, w \rangle \mid M \text{ is a DFA that accepts } w^n \text{ for all } n \geq 0\}.$$

Show that $L \in \mathbf{P}$ by giving a polynomial-time decision procedure for L. (A high-level algorithm will suffice.)

Answer: Let

N = "On input $\langle M, w\rangle$ where M is a DFA and w is a string:

(a) Let n be the number of states of M

(b) For $i := 0$ to $n-1$ do

  i. Run M on input w^i

  ii. If M rejects w^i, then reject

(c) Accept"

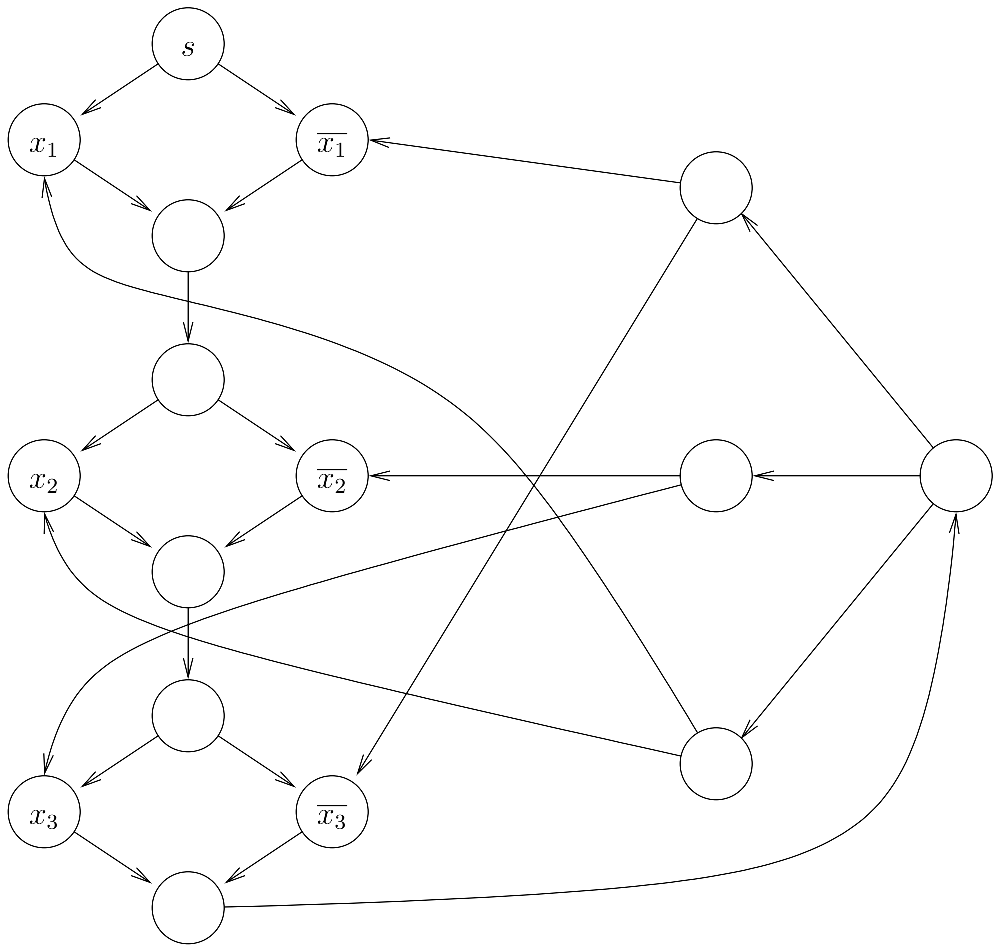
Why this works: Clearly, N runs in polynomial time (quartic time, actually, although this is not optimal). It is also clear that if $\langle M, w\rangle \in L$ then N accepts $\langle M, w\rangle$. But the converse is also true: Suppose $M = (Q, \Sigma, \delta, q_0, F)$, where $|Q| = n$. Let $q_0, q_1, q_2, \ldots$ be the sequence of states that M enters after reading inputs $w^0, w^1, w^2, \ldots$, respectively. That is, let $q_i = \delta(q_0, w^i)$ for all $i \geq 0$, and note that $q_{i+1} = \delta(q_i, w)$. By its construction, N accepts $\langle M, w\rangle$ if and only if $\{q_0, q_1, \ldots, q_{n-1}\} \subseteq F$. Since M has only n states, by the Pigeon Hole Principle, there must be some $0 \leq i < j \leq n$ such that $q_i = q_j$ (this is similar to the proof of the Pumping Lemma). But then, $q_{i+1} = q_{j+1}$, $q_{i+2} = q_{j+2}$, etc. That is, we have a loop, which implies that all states in the sequence are in the set $\{q_0, q_1, \ldots, q_{n-1}\}$. Thus N only needs to test whether these states are all in F.

7. (20 points total) Let φ be the quantified Boolean formula

$$(\exists x_1)(\forall x_2)(\exists x_3)[\ (x_1 \vee x_3) \wedge (x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2})\ ].$$

(a) (15 points) Give the instance $\langle G, s\rangle$ of GG that φ maps to via the p-m-reduction of TQBF to GG described either in the book or in class. (Draw the digraph G and label the start vertex s.)

Answer:



(b) (5 points) Is φ true?

Answer: No, φ is false. For any truth value Alice picks for x_1, Bob picks the same truth value for x_2, which guarantees that at least one clause is violated, regardless of Alice's choice of x_3. This shows that Bob has a winning strategy, not Alice.