Answers to the CSCE 551 Final Exam, April 30, 2008

1. (15 points) Use the Pumping Lemma to show that the language

\[ L = \{ x \in \{0, 1\}^* \mid \text{the number of 0s and 1s in } x \text{ differ (in either direction) by at most } 2008 \} \]

is not regular.

**Answer:** Given \( p > 0 \), let \( s = 0^p 1^{p+2008} \). Clearly, \( |s| = 2p + 2008 \geq p \) and \( s \in L \).

If \( x, y, z \) are such that \( xyz = s \), \( |xy| \leq p \), and \( |y| > 0 \), then we must have \( y = 0^m \) for some \( m > 0 \). Letting \( i = 0 \), we remove \( m \) zeros to get

\[ xy^i z = x y^0 z = xz = 0^{p-m} 1^{p+2008} \notin L. \]

Thus by the Pumping Lemma, \( L \) is not regular. (There are many other workable choices of \( s \) and \( i \).)

2. (10 points) The following DFA is not minimum:

![DFA Diagram](image)

Using any method you like, find the minimum equivalent DFA. (Draw its transition diagram.)

**Answer:** States \( A \) and \( C \) are the only pair of indistinguishable states. Combining them, we get the minimum DFA:
3. (15 points) Give an m-reduction from $A_{\text{TM}}$ to the language

$$B = \{\langle R \rangle \mid R \text{ is a TM that accepts a string } w \text{ iff } |w| = 2008 \}.$$ 

**Answer:** Let

$$f = \text{“On input } \langle M, w \rangle \text{ where } M \text{ is a TM and } w \text{ is a string:}$$

(a) Let $R = \text{‘On input } x:\$

(i) If $|x| \neq 2008$ then reject

(ii) Simulate $M$ on $w$ (and do what $M$ does)’

(b) Output $\langle R \rangle$”

Then $f$ m-reduces $A_{\text{TM}}$ to $B$.

4. (25 points total) Fix some enumerator $E$. Let $A$ be the language of all strings $w$ such that

- $E$ prints $w$ at least once, and
- $w$ is at least as long as any string that $E$ prints before it first prints $w$.

Show:

(a) (15 points) $A$ is decidable.

(b) (10 points) If $E$ enumerates an infinite language, then $A$ is infinite.

[This proves that every infinite Turing recognizable set has an infinite decidable subset.]
Let $L$ be the language enumerated by $E$. It follows from the definition that $A \subseteq L$.

(a) There are two cases:

$L$ is finite: Then $A$ is also finite. Since every finite language is decidable (regular even), then $A$ is decidable.

$L$ is infinite: Let $D$ implement the following algorithm:

"On input $w$:
  i. Run $E$ until it either prints $w$ or it prints some string longer than $w$, whichever comes first
  ii. If $E$ prints $w$, then accept
  iii. If $E$ prints a string longer than $w$, then reject"

Since $E$ enumerates an infinite language (by assumption), it must print strings that are arbitrarily long. Thus on any input $w$, Step (i) will eventually complete, and so $D$ is a decider. It is also straightforward to see that $w \in A$ iff $D$ accepts $w$. Thus $D$ decides $A$, and so $A$ is decidable.

[Note that in general we cannot compute, given $\langle E \rangle$, which case holds—finite or infinite. This question is undecidable, but it does not affect the decidability of $A$.]

(b) Assume that $L$ is infinite. For any length $n \geq 0$, there are only finitely many strings of length $\leq n$, and so $E$ must eventually print a string longer than $n$. The first such string printed by $E$ is in $A$ (clearly). So $A$ contains a string longer than $n$. Since $n$ is arbitrary, $A$ must contain arbitrarily long strings, so $A$ must be infinite.

5. (10 points) Prove that if TQBF $\in$ NP, then NP $=$ PSPACE.

Answer: We already know that NP $\subseteq$ PSPACE (this was proven in class), so it suffices to show that PSPACE $\subseteq$ NP. Let $A$ be any language in PSPACE. We know that TQBF is PSPACE-complete, so in particular, TQBF is PSPACE-hard, which means $A \leq^p_m$ TQBF because $A \in$ PSPACE. Now suppose that TQBF $\in$ NP. We proved in class that for any languages $C$ and $D$, if $C \leq^p_m D$ and $D \in$ NP, then $C \in$ NP. In this case, we have $A \leq^p_m$ TQBF and TQBF $\in$ NP (by assumption), and thus $A \in$ NP. Since $A$ was any PSPACE language, this shows that PSPACE $\subseteq$ NP.

6. (15 points) Let

$$L = \{ \langle M, w \rangle \mid M \text{ is a DFA that accepts } w^n \text{ for all } n \geq 0 \}.$$

Show that $L \in P$ by giving a polynomial-time decision procedure for $L$. (A high-level algorithm will suffice.)
Answer: Let

\[ N = \text{"On input } \langle M, w \rangle \text{ where } M \text{ is a DFA and } w \text{ is a string:} \]

(a) Let \( n \) be the number of states of \( M \)
(b) For \( i := 0 \) to \( n - 1 \) do
   i. Run \( M \) on input \( w^i \)
   ii. If \( M \) rejects \( w^i \), then reject
(c) Accept

Why this works: Clearly, \( N \) runs in polynomial time (quartic time, actually, although this is not optimal). It is also clear that if \( \langle M, w \rangle \in L \) then \( N \) accepts \( \langle M, w \rangle \).

But the converse is also true: Suppose \( M = (Q, \Sigma, \delta, q_0, F) \), where \( |Q| = n \). Let \( q_0, q_1, q_2, \ldots \) be the sequence of states that \( M \) enters after reading inputs \( w^0, w^1, w^2, \ldots \), respectively. That is, let \( q_i = \delta(q_0, w^i) \) for all \( i \geq 0 \), and note that \( q_{i+1} = \delta(q_i, w) \).

By its construction, \( N \) accepts \( \langle M, w \rangle \) if and only if \( \{q_0, q_1, \ldots, q_{n-1}\} \subseteq F \). Since \( M \) has only \( n \) states, by the Pigeon Hole Principle, there must be some \( 0 \leq i < j \leq n \) such that \( q_i = q_j \) (this is similar to the proof of the Pumping Lemma). But then, \( q_{i+1} = q_{j+1}, q_{i+2} = q_{j+2}, \) etc. That is, we have a loop, which implies that all states in the sequence are in the set \( \{q_0, q_1, \ldots, q_{n-1}\} \). Thus \( N \) only needs to test whether these states are all in \( F \).

7. (20 points total) Let \( \varphi \) be the quantified Boolean formula

\[ (\exists x_1)(\forall x_2)(\exists x_3)[(x_1 \lor x_3) \land (x_2 \lor \overline{x_3}) \land (\overline{x_1} \lor \overline{x_2})]. \]

(a) (15 points) Give the instance \( \langle G, s \rangle \) of GG that \( \varphi \) maps to via the p-m-reduction of TQBF to GG described either in the book or in class. (Draw the digraph \( G \) and label the start vertex \( s \).)

Answer:
(b) (5 points) Is $\varphi$ true?

**Answer:** No, $\varphi$ is false. For any truth value Alice picks for $x_1$, Bob picks the same truth value for $x_2$, which guarantees that at least one clause is violated, regardless of Alice’s choice of $x_3$. This shows that Bob has a winning strategy, not Alice.