CSCE 355
3/25/2024

Today: PDA $\longrightarrow$ CFG

Theorem: For every PDA P there exists a CFG G such that $N(P) = L(G)$.

Proof: By construction. — {WLOG: only care about restricted PDAs}

Given a restricted PDA $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ (ignored)

Let $w \in N(P)$ be a string accepted by P and let

$$p := ID_0 \vdash ID_1 \vdash \cdots \vdash ID_n$$ be an accepting computation of w

$ID_0 = (q_0, w, Z_0)$, $ID_n = (q, \varepsilon, \varepsilon)$ [some $q \in Q$]

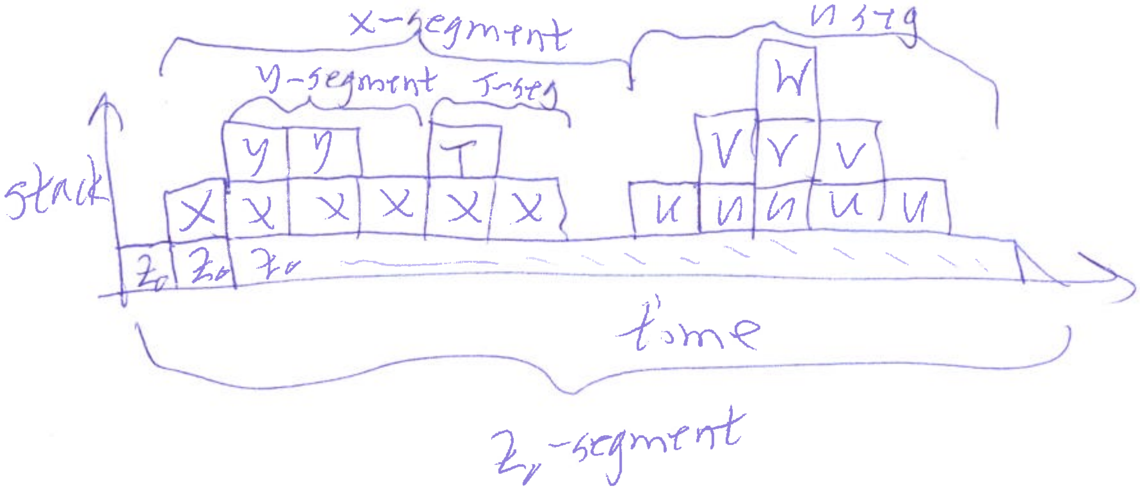
Let $X \in \Gamma$ be a stack symbol.

An X-segment of p is a ~~step~~ subsequence of p

$$ID_i \vdash ID_{i+1} \vdash \cdots \vdash ID_j \quad \text{some } j > i$$

such that X appears on top of the stack at ID_i, stays on the stack through ID_{j-1}, but gets popped off going from ID_{j-1} to ID_j

Note: p is a Z_0-segment.



Note that a string w is in $N(P)$ iff there is a computation reading all of w that forms a Z_0-segment, starting in state q_0.

To define $G = \langle V, \Sigma, S, P \rangle$

$$V := \{ [qXr] : q, r \in Q, X \in \Gamma \} \cup \{ S \}$$

Idea: $[qXr]$ will derive all possible strings read during an X segment that starts in state q and ends in state r.

Productions of G:

1. For every $q \in Q$ add the production

$$S \to [q_0 Z_0 q]$$

2. For every $q, r \in Q$ and $X \in \Gamma$ and $a \in \Sigma \cup \{\varepsilon\}$ such that

$$(r, pop) \in \delta(q, a, X)$$

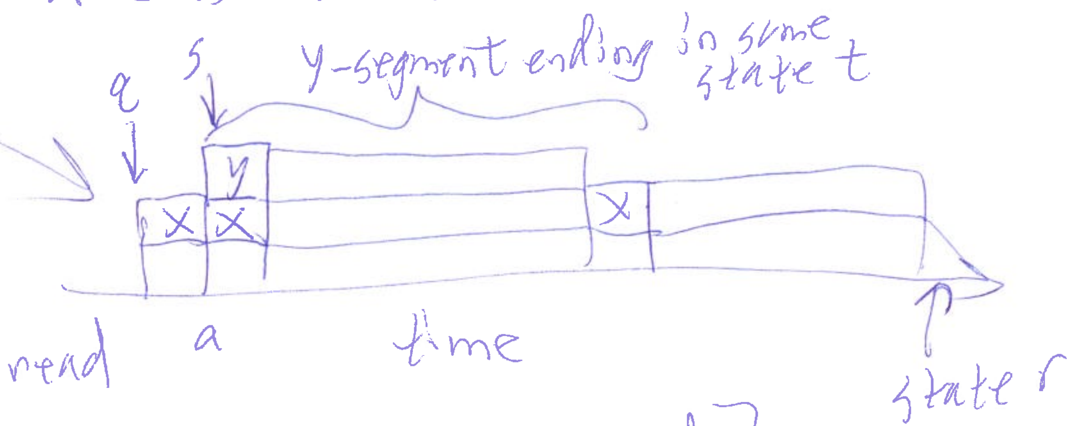
add the production

$$[qXr] \to a$$

3. For every $q, r \in Q$, $X \in \Gamma$, and $a \in \Sigma \cup \{\varepsilon\}$, and every state s, such that $\delta(q,a,X)$ contains $(s, \text{push } Y)$ add the productions

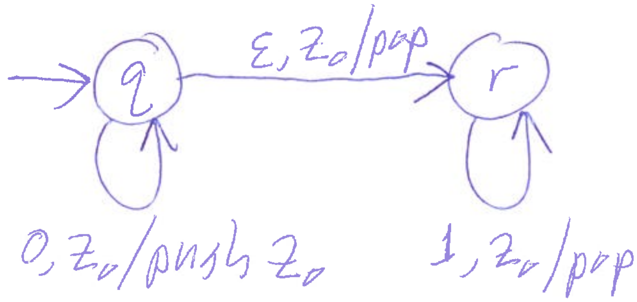
$$[qXr] \to a[sYt][tXr] \quad \text{for all } t \in Q$$

End of construction



[Proof of correctness omitted.]

Example: Restricted PDA recognizing $\{0^n 1^n : n \geq 0\}$



$\Gamma = \{Z_0\}$

4

Grammar: $V = \{S, qZ_0q, qZ_0r, rZ_0q, rZ_0r\}$

S-productions $S \to [qZ_0q] \mid [qZ_0r]$

popping productions: $[qZ_0r] \to \varepsilon$

$[rZ_0r] \to 1$

pushing productions $[qZ_0q] \to 0[qZ_0q][qZ_0q] \mid 0[qZ_0r][rZ_0q]$

$[qZ_0r] \to 0[qZ_0q][qZ_0r] \mid 0[qZ_0r][rZ_0r]$

Not optimal! Simplify:

$A := [qZ_0q]$; $B := [qZ_0r]$; $C := [rZ_0q]$; $D := [rZ_0r]$

rewrite the grammar:

$S \to \cancel{A} \mid B$

$B \to \varepsilon$

$\cancel{D \to 1}$

$\cancel{A \to 0AA} \mid \underbrace{\cancel{0BC}}_{\text{useless — can't get rid of C}}$

$B \to \cancel{0AB} \mid \cancel{0BD}$

$0B1$

$S \to B$
$B \to \varepsilon$
$B \to 0B1$

$S \to \varepsilon$
$S \to 0S1$

Two more topics about CFLs:

- Closure properties of CFLs
- Pumping Lemma for CFLs

Already know: CFLs are closed under

union : L_1, L_2 CFLs $\Rightarrow$ $L_1 \cup L_2$ is a CFL

concatenation: L_1, L_2 CFLs $\Rightarrow$ $L_1 L_2$ is a CFL

CFLs are <u>not</u> closed under complement:

Counter-example:

$L := \{w \in \{0,1\}^* : w$ is not of the form xx for any $x\}$

is a CFL but

$\overline{L} := \{xx : x \in \{0,1\}^*\}$ is <u>not a CFL</u>

[pumping lemma for CFLs]

PDA P that recognizes L:

P has a branch that accepts if $|w|$ is odd

Assume $|w|$ is even then.