CSCE 355
3/13/2024

PDAs (continued)

Recall: An ID of a PDA $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ is a triple (q, x, γ)

- q — current state
- x — unconsumed remainder of the input
- γ — current stack contents

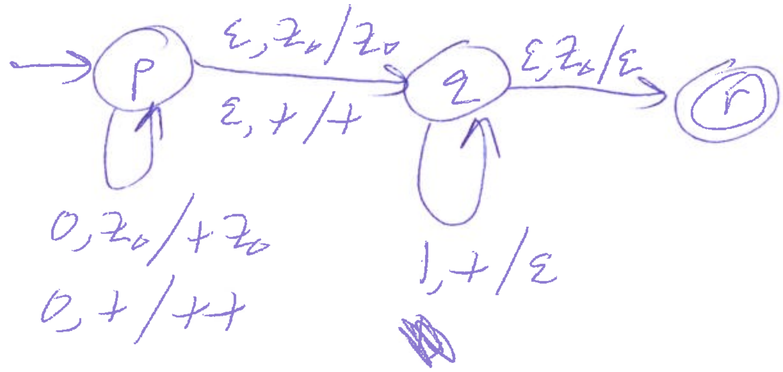
$ID \vdash ID'$ — 1-step successor

$ID \vdash^* ID'$ means $ID = ID_0 \vdash ID_1 \vdash \cdots \vdash ID_n = ID'$ for some $n \geq 0$ and IDs $ID_1, \ldots, ID_{n-1}$

In particular, $ID \vdash^* ID$ (0 steps)

Recall:

recognizes $\{0^n 1^n : n \geq 0\}$



Input is 0011

$(p, 0011, Z_0) \vdash (p, 011, +Z_0) \vdash (p, 11, ++Z_0) \vdash (q, 11, ++Z_0) \vdash (q, 1, +Z_0)$

$(p, 0011, Z_0) \vdash (q, 0011, Z_0) \vdash (r, 0011, \varepsilon)$ X

$(p, 011, +Z_0) \vdash (q, 011, +Z_0)$ X

$(q, 1, +Z_0) \vdash (q, \varepsilon, Z_0) \vdash (r, \varepsilon, \varepsilon)$ — accepting

Def of acceptance (revised from last class)

Let P be a PDA as above and $w \in \Sigma^*$ a string.

1. P accepts w by final state if

$$(q_0, w, Z_0) \vdash^* (r, \varepsilon, \gamma) \text{ for some } \gamma \in \Gamma^* \text{ and } r \in F$$

[defined last class]

2. P accepts w via empty stack if

$$(q_0, w, Z_0) \vdash^* (q, \varepsilon, \varepsilon) \text{ for some } q \in Q.$$

$L(P) := \{w \in \Sigma^* : P \text{ accepts } w \text{ via final state}\}$

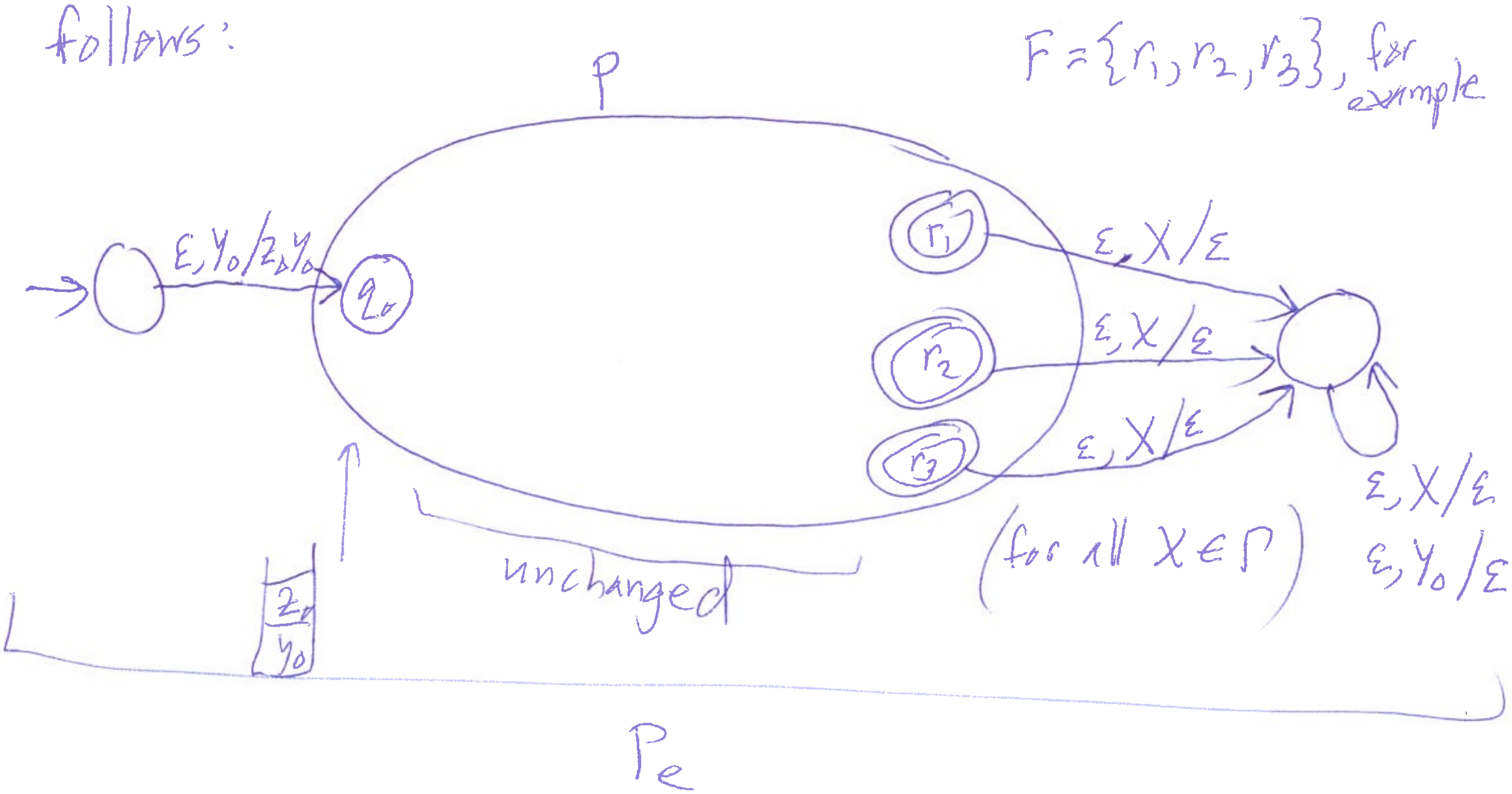
$N(P) := \{w \in \Sigma^* : P \text{ accepts } w \text{ by empty stack}\}$

Coincidentally, $L(P) = N(P)$ for P being the previous example, but this is not generally true.

Theorem: Let $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA. There exist PDAs P_e, P_f such that

1. $L(P) = N(P_e)$ and
2. $N(P) = L(P_f)$

Proof: For (1): We construct P_e from P as follows:

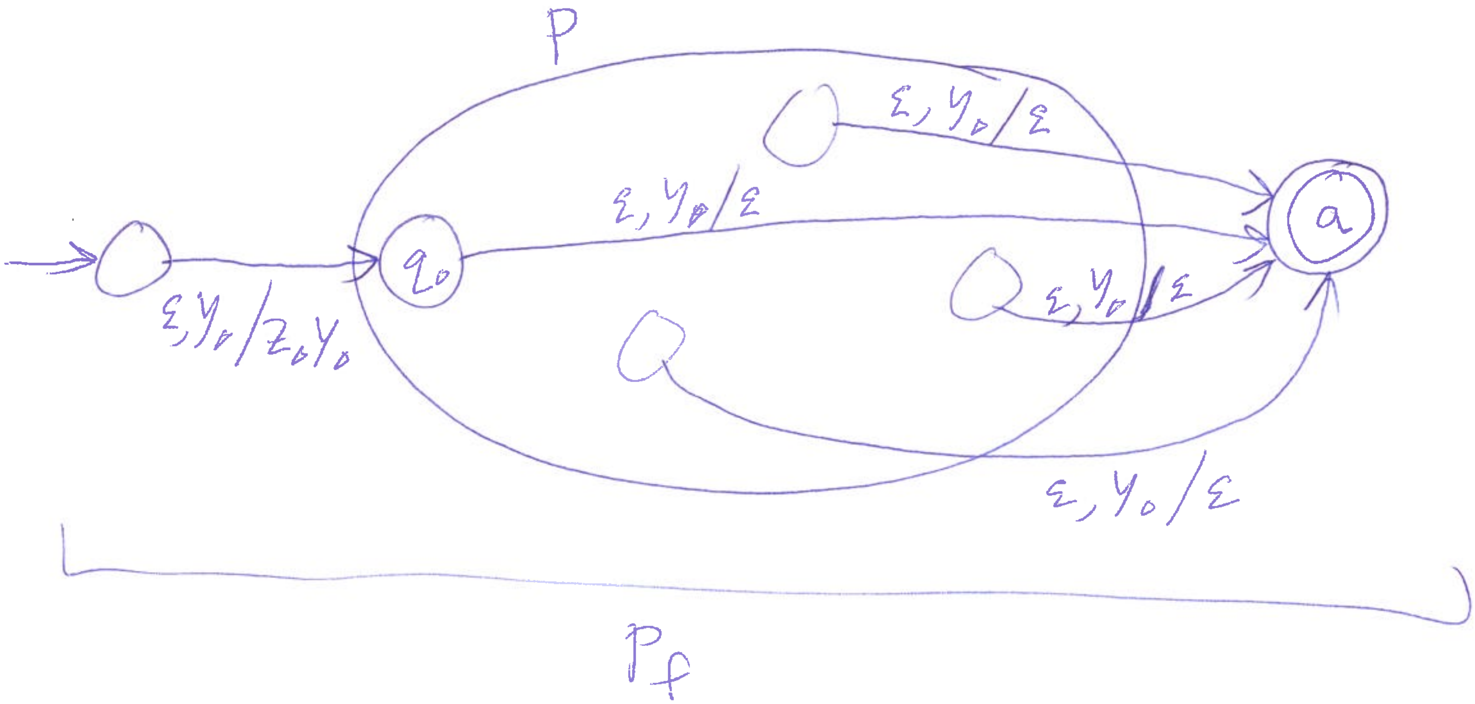


- P_e's bottom stack marker is $Y_0 \notin \Gamma$ (not one of P's stack symbols)

$\therefore N(P_e) = L(P)$

[Proof of correctness omitted]

For (2) we construct P_f such that $L(P_f) = N(P)$ as follows:



P_f has its own bottom stack marker $Y_0 \notin \Gamma$

- Add $\varepsilon, Y_0/\varepsilon$ transition from every state of P to new accepting state a of P_f.
- Make all accepting states of P rejecting

Now $L(P_f) = N(P)$
[$= N(P_f)$]

[Proof of correctness omitted]

∎

Now: CFG $\Longrightarrow$ PDA
(1-state!)

Theorem: For every CFG G there exists a PDA P such that $L(G) = N(P)$.

Proof: Let $G := \langle V, \Sigma, S, Prod \rangle$.

Let
$$P := \langle \{q\}, \underbrace{\Sigma}_{\text{input alphabet}}, \underbrace{\Sigma \cup V}_{\text{stack alphabet}}, \delta, q, S, \emptyset \rangle$$

where δ is defined as follows

For every $a \in \Sigma$, have
$$\delta(q, a, a) = \{(q, \varepsilon)\}$$
(matching transitions)

For every production $A \to \beta \in Prod$, we have
$$\delta(q, \varepsilon, A) \ni (q, \beta)$$
($\ni$: contains)

No other transitions

Example:

G:
$S \to 0S1$
$S \to \varepsilon$

$L(G) = \{0^n 1^n : n \geq 0\}$

$P = \langle \{q\}, \{0,1\}, \{0,1,S\}, \delta, q, S, \emptyset \rangle$

where

$\delta(q,0,0) = \{(q,\varepsilon)\}$
$\delta(q,1,1) = \{(q,\varepsilon)\}$ — matching transitions

$\delta(q,\varepsilon,S) = \{(q,0S1), (q,\varepsilon)\}$

~~[δ]~~ No other transitions

Input 0011

$(q,0011,S) \vdash (q,0011,0S1) \vdash (q,011,S1)$

initial ID; expand on $S \to 0S1$; match 0

$\vdash (q,011,0S11)$

match 0 → $\vdash (q,11,S11) \vdash (q,11,11)$

expand on $S \to \varepsilon$

$\vdash (q,1,1)$

match 1 → $\vdash (q,\varepsilon,\varepsilon)$ accept!