CSCE 355
2/21/2024

Pumping Lemma: Every regular language is pumpable.

Recall: A lang. $L$ is pumpable iff

$\exists p > 0$ (the pumping length),

$\forall s$, $s \in L$ & $|s| \geq p$,

$\exists x, y, z$ (strings), $s = xyz$, $|xy| \leq p$, $y \neq \varepsilon$,

$\forall i \geq 0$,

$xy^iz \in L$

$L$ is not pumpable [$\therefore$ not regular] iff

$\forall p > 0$

$\exists s$, $s \in L$ & $|s| \geq p$

$\forall x, y, z$ such that $s = xyz$, $|xy| \leq p$, $y \neq \varepsilon$,

$\exists i \geq 0$,

$xy^iz \notin L$

Example: $L := \{0^n 1^n : n \geq 0\}$ is not pumpable.

**Proof:** Given any $p > 0$,

let $s := \underline{0}^p 1^p$.

$[s \in L \ \& \ |s| = 2p \geq p]$

Given $x, y, z$ such that $s = xyz$, $|xy| \leq p$, $y \neq \varepsilon$,

let $i := 0$.

Then $xy^iz = xy^0z = xz \notin L \leftarrow$ claim.

Claim that $xz \notin L$. Since $s = xyz$ & $|xy| \leq p$,

then $xy$ only contains $0$'s.

Then Since $y \neq \varepsilon$, $y = 0^k$ for some $k > 0$.

(positive $k$)

But then $xz = 0^{p-k}1^p \notin$ since $p-k \neq p$.

$\therefore$ Thus $L$ is not pumpable, hence not regular (by the Pumping Lemma). //

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Proof template for nonpumpability:

"Given $p > 0$, let $s :=$ ________ $[s \in L \ \& \ |s| \geq p]$.

Then, given any $x, y, z$ such that $s = xyz$, $|xy| \leq p$, $y \neq \varepsilon$

let $i :=$ ________.

Then $xy^iz \notin L$ because ________."

Ex: $L = \{w \in \{0,1\}^* : w$ has equal number of 0's and 1's$\}$

Prop: $L$ is not pumpable.

Proof: Given $p > 0$, let $s := 0^p 1^p$. $[s \in L$ & $|s| \geq p]$

Then given $x, y, z$ such that $s = xyz$, $|xy| \leq p$, $y \neq \varepsilon$,

let $i := 0$.

Then $xz \notin L$ because $y = 0^k$ for some $k > 0$,

so $xz = 0^{p-k} 1^p \notin L$ because unequal numbers of 0's and 1's.

$\therefore L$ is not pumpable. //

Ex: $L := \{0^m 1^n : 0 \leq m \leq n\}$

Prop: $L$ not pumpable.

Proof: Given $p > 0$, let $s := 0^p 1^p$ $[s \in L$ & $|s| \geq p]$

Given $x, y, z$ such that ....

Let $i := 2$ (or anything greater).

Then $xyyz \notin L$ because $y = 0^k$ for some $k > 0$,

so $xyyz = 0^{p+k} 1^p \notin L$ because $p + k > p$.

$\therefore L$ not pumpable. //

**Ex:** $L := \{w \in \{0,1\}^* : |w|$ is even and no 0's in the 1st half of $w\}$

**Prop:** $L$ not pumpable.

**Proof:** Given $p > 0$, let $s := 1^p 0^p$ $[s \in L$ & $|s| \geq p]$

Given $x, y, z$ ~ ~ ~ ~

Let $i := 0$.

Then $xz \notin L$ since $y = 1^k$ for some $k > 0$,

so $xz = 1^{p-k} 0^p$, and this string either has odd length (if $k$ is odd) or has a 0 in its 1st half. Either way, $xz \notin L$.

$\therefore$ $L$ not pumpable //

**Prop:** $L := \{ \cancel{w \in \{0,1\}^* :}\ 0^m 1^n : 0 \leq m \leq n\}$

**Bad proof:** Given $p > 0$, let $s := 0^{p-1} 1^p$

 lets $x := 0^{p-1}$

$y := 1$

$z := 1^{p-1}$

☹ — no choice of $i \geq 0$ gives $xy^iz \notin L$.

~~Another bad~~ A good proof proof:

Given $p > 0$, let $s := 0^p 1^{p+1}$

Given $x, y, z$,

let $i := \begin{cases} 2 & \text{if } y = 0^k \text{ for } k \geq 2 \\ 3 & \text{otherwise} \end{cases}$

[Actually $i := 3$ works regardless]

Ex: $L := \{ \cancel{w \in \{0,1\}^* :} \ 0^m 1^n : 0 \leq n \leq m \}$

Bad proof of nonpumpability:

Given $p > 0$, let $s := \underline{0^p 1^{p-1}}$ bad choice

Given $x, y, z$ [$y = 0^k$ some $k > 0$]

Adversary chooses $x := 0^{p-1}$

$y := 0$

$z := 1^{p-1}$

Any choice of $i$ removes $\leq 1$ many 0's,

so $xy^iz \in L$ for all $i \geq 0$.

Good choice for $s$ is $0^p 1^p$ because it is on the "edge" of the language

[$i := 0$ now works].

Ex: $L := \{0^m 1^n : m \neq n\}$

~~Proof~~ Prop: $L$ is not regular.

Proof: (by contradiction). Assume $L$ is regular.

Then $\overline{L}$ is regular (reg. lang's closed under complement)

But then $\overline{L} \cap \underbrace{L(0^* 1^*)}_{\text{regular}}$ is regular

(reg. lang's are closed under intersection).

But $\overline{L} \cap L(0^* 1^*)$

$$= \overline{L} \cap \{0^m 1^n : m, n \geq 0\}$$

$$= \{0^m 1^n : 0 \leq m = n\} = \underbrace{\{0^n 1^n : n \geq 0\}}_{\substack{\text{not regular b/c} \\ \text{not pumpable} \\ \text{(Pumping Lemma)}}} \;↯$$

$\therefore$ $L$ not regular. ∎