NOTE the following definitions:

- The binary alphabet is the set \{0, 1\}.
- A binary string is any string over the binary alphabet.
- If \(w\) is any string, then \(w^R\) (the reversal of \(w\)) is \(w\) written backwards, that is, comprising the symbols of \(w\) in reverse order.
- A string \(x\) is a prefix of a string \(y\) iff there exists a string \(z\) such that \(y = xz\).
- A string \(x\) is a suffix of a string \(y\) iff there exists a string \(z\) such that \(y = zx\).
- A string \(x\) is a substring of a string \(y\) if there exist strings \(w, z\) such that \(y = wxz\).

1. Consider the following DFA:

   ![DFA Diagram](image)

   (a) For each of the strings below, say which state the DFA is in after reading the string, and say whether or not the DFA accepts the string.

   \[aaa \quad bb \quad bbb \quad abab \quad bbbbbbbbbbbbbbaaa \quad \varepsilon \quad aabbbbababbaaaabbaabbababbbb\]

   (b) Give two different strings of length 4 that each make the DFA go from state 0 to state 1.

2. Draw a DFA with alphabet \(\{0, 1\}\) that accepts a binary string \(x\) iff \(x\) has odd length, i.e., iff \(|x|\) is odd.
3. Let $A$ be the DFA given by the following tabular form:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\to *q_0$</td>
<td>$q_0$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_0$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$q_1$</td>
<td>$q_2$</td>
</tr>
</tbody>
</table>

($A$ accepts a binary string iff it represents a multiple of 3.) Recall the DFA described in class (here we’ll call it $B$) that accepts a binary string iff the string ends with 1:

Recall the product construction from class. Draw the diagram for the product of $A$ and $B$ so the resulting DFA recognizes the language $L(A) \cap L(B)$.

4. Describe a DFA $B$ that accepts a string over the alphabet \{a, b, c\} iff its first and last symbols are different.

5. Consider the following two languages over the alphabet \{a, b\}:

   $L_1 = \{w \mid w$ is either the empty string or ends with b\}$
   $L_2 = \{w \mid$ there is a b followed by an a somewhere in $w$\}$.

   (a) Draw a 2-state DFA recognizing $L_1$ and a 3-state DFA recognizing $L_2$.
   (b) Using your answer and the product construction, draw a DFA recognizing $L_1 \cap L_2$.
       Do not perform any optimizations (e.g., removing unreachable states or transitions, or merging indistinguishable states).

6. Give the transition diagram for a DFA over the alphabet $\Sigma = \{a, b, c\}$ that accepts a string $w$ iff $w$ contains $ab$ as a substring but does not contain $abb$ as a substring. What is the least number of states you need?

7. (Optional) This exercise is adapted from Exercise 2.2.1 on pages 52–53, which is formulated somewhat vaguely. Consider the marble-rolling toy (redrawn from Figure 2.8):
A marble is dropped at $A$ or $B$. Levers $x_1$, $x_2$, and $x_3$ cause the marble to fall either to the left or to the right. whenever a marble encounters a lever, it causes the lever to reverse after the marble passes, so the next marble will take the opposite branch.

Model this toy as a finite automaton. An input to the automaton is a string over the alphabet \{A, B\}, which represents a sequence of marbles being dropped into the toy. The toy is initially in the configuration above before any marbles are dropped (so that the first ball will exit at $C$ regardless of where it is dropped). Say that a sequence of marble drops is accepted exactly in the case that if one additional marble were to be dropped in, it would go out through $D$ regardless of where it was dropped.