CSCE 355, Answer Key to Assignment 1

1 Definitions

Unchanged from the Homework 1 problem set.

2 Problems

2.1 Sets

1. Let $A := \{1, 2, 3, 4\}$ and $B := \{2, 5\}$. What are (a) $A \cup B$, (b) $A \cap B$, (c) $A - B$, and (d) $A \triangle B$? What are (e) $A \times B$ and (f) 2^B? In each case, also give the cardinality of the set.

   Answer:

   (a) $A \cup B = \{1, 2, 3, 4, 5\}$. $|A \cup B| = 5$.
   (b) $A \cap B = \{2\}$. $|A \cap B| = 1$.
   (c) $A - B = \{1, 3, 4\}$. $|A - B| = 3$.
   (d) $A \triangle B = \{1, 3, 4, 5\}$. $|A \triangle B| = 4$.
   (e) $A \times B = \{(1, 2), (1, 5), (2, 2), (2, 5), (3, 2), (3, 5), (4, 2), (4, 5)\}$. $|A \times B| = 8$.
   (f) $2^B = \{\emptyset, \{2\}, \{5\}, \{2, 5\}\}$. $|2^B| = 4$.

2. True or false: $2^\emptyset = \emptyset$. Explain.

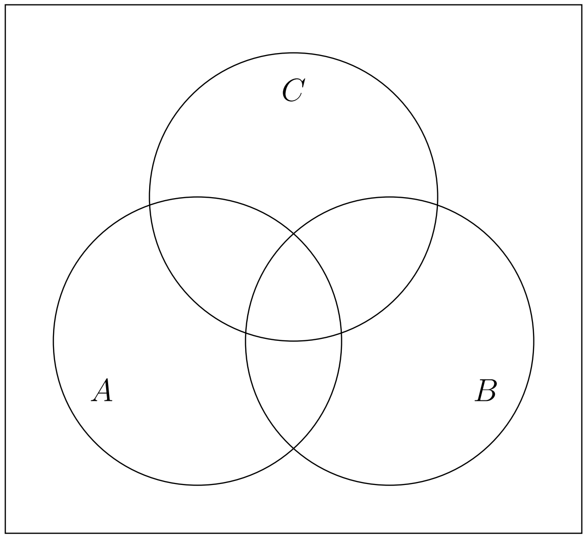
   Answer: False. $2^\emptyset = \{\emptyset\} \neq \emptyset$. That is, $2^\emptyset$ is not empty; it contains the element $\emptyset$, because $\emptyset \subseteq \emptyset$.

3. Using just braces and commas, write the set $2^{2^{\{\emptyset\}}}$ in "long hand."

   Answer:

$$\begin{aligned} 2^{2^{\{\emptyset\}}} &= 2^{\{\emptyset, \{\emptyset\}\}} \\ &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\ &= \{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\}, \{\{\}\}\}\} \,. \end{aligned}$$

4. Using the figure shown below as a template, draw and fill in a Venn diagram to illustrate each of the expressions below involving sets A, B, C. That is, shade the regions that are part of the expression (one Venn diagram per expression):



(a) $A \cap B \cap C$

(b) $A \cup B \cup C$

(c) $A \cap (B \cup C)$

(d) $A - (B \cap C)$

(e) $(A \cup B) - C$

(f) $A - (B - C)$

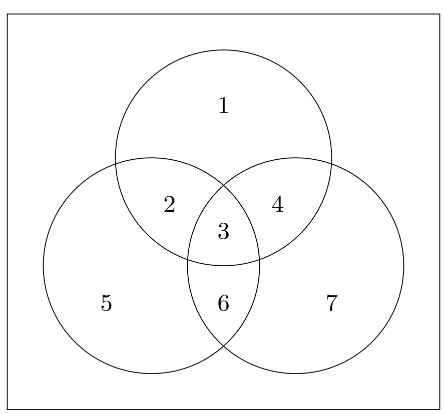
(g) $(A \triangle B) \triangle C$

(h) $(A \cap B) \cup (A \cap C)$

(i) $A \triangle (B \triangle C)$

(j) $A \triangle (B \cap C)$

Answer:



expression	shaded regions
$A \cap B \cap C$	3
$A \cup B \cup C$	$1, 2, 3, 4, 5, 6, 7$
$A \cap (B \cup C)$	$2, 3, 6$
$A - (B \cap C)$	$2, 5, 6$
$(A \cup B) - C$	$5, 6, 7$
$A - (B - C)$	$2, 3, 5$
$(A \triangle B) \triangle C$	$1, 3, 5, 7$
$(A \cap B) \cup (A \cap C)$	$2, 3, 6$
$A \triangle (B \triangle C)$	$1, 3, 5, 7$
$A \triangle (B \cap C)$	$2, 4, 5, 6$

5. What set theoretic identities holding for all A, B, C are shown by your Venn diagrams in the last problem?

   Answer: The last problem illustrates the following identities, which hold for all sets A, B, C:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \ ,$$
$$(A \triangle B) \triangle C = A \triangle (B \triangle C) \ .$$

6. Show that the symmetric difference operation $\triangle$ on sets is commutative and associative, and that $A \triangle A = \emptyset$ for all sets A.

   Answer: Using the Venn diagram above we see that $A \triangle B$ and $B \triangle A$ both shade regions $5, 7$. Thus the two expressions are equal, showing commutativity. $(A \triangle B) \triangle C$ and $A \triangle (B \triangle C)$ both shade regions $1, 3, 5, 7$, showing that the two expressions are equal, thus associativity. Finally, $A \triangle A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$.

2.2 Relations

Let $A := \{1, 2, 3, 4\}$ as in the first problem, above.

7. Let $R := \{(1, 2), (2, 3), (3, 4)\}$. ($R$ is a relation on A.)

   (a) Add the fewest possible ordered pairs to R to make a reflexive relation (on A).

   (b) Add the fewest possible ordered pairs to R to make a symmetric relation.

   (c) Add the fewest possible ordered pairs to R to make a transitive relation.

   (d) Add the fewest possible ordered pairs to R to make an equivalence relation on A. How many equivalence classes are there?

   Answer:

   (a) Add $(1, 1), (2, 2), (3, 3), (4, 4)$.

   (b) Add $(2, 1), (3, 2), (4, 3)$.

   (c) Add $(1, 3), (1, 4), (2, 4)$.

   (d) We must add all the other possible ordered pairs, i.e., all those pairs in $A \times A$ that are not already in R (13 pairs in all). This will make a single equivalence class.

8. Same as the last problem, but now let $R := \{(1, 2), (2, 3), (3, 1), (4, 4)\}$.

   Answer:

   (a) Add $(1, 1), (2, 2), (3, 3)$.
   (b) Add $(2, 1), (3, 2), (1, 3)$.
   (c) Add $(1, 1), (1, 3), (2, 1), (2, 2), (3, 2), (3, 3)$.
   (d) Add $(1, 1), (1, 3), (2, 1), (2, 2), (3, 2), (3, 3)$ (the same pairs as in the last problem). This makes two equivalence classes: $\{1, 2, 3\}$ and $\{4\}$.

9. Give an example of a nonempty binary relation on A that is symmetric and transitive but not reflexive.

   Answer: A minimum size answer is $\{(1, 1)\}$. Another answer is $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Other answers are possible.

10. Suppose $\leq$ is a quasiorder on some set A. For every $a, b \in A$, define

$$a \equiv b \iff (a \leq b \text{ and } b \leq a) .$$

   Show that $\equiv$ is an equivalence relation on A.

   Answer: Let a, b, c be any elements of A. To prove reflexivity of $\equiv$, we have $a \leq a$ by reflexivity of $\leq$, and thus $a \equiv a$. For symmetry, suppose $a \equiv b$. Then $a \leq b$ and $b \leq a$, which equivalent to $b \equiv a$. For transitivity, suppose $a \equiv b$ and $b \equiv c$. Then $a \leq b$ and $a \leq c$, and thus $a \leq c$ by transitivity of $\leq$; similarly, we have $c \leq b$ and $b \leq a$, and so $c \leq a$—again by transitivity of $\leq$. We conclude that $a \equiv c$, which shows transitivity of $\equiv$. Thus $\equiv$ is an equivalence relation, since it is reflexive, symmetric, and transitive.

2.3 Functions

Let A and B be as in the first problem, above.

11. Give an example of a one-to-one function $f : B \to A$. How may such functions are there?

    Answer: The function $f := \{(2, 1), (5, 2)\}$ is one-to-one from B to A. There are exactly $4 \times 3 = 12$ such functions, as one has 4 choices of where to map 2, and for each of these choices, 3 remaining choices of where to map 5.

12. Give an example of an onto function $g : A \to B$. How may such functions are there?

    Answer: The function $g := \{(1, 2), (2, 2), (3, 5), (4, 5)\}$ maps A onto B. There are exactly $16 - 2 = 14$ such functions. (There are $2^4 = 16$ functions in all, but two of these are not onto: the function mapping everything to 2 and the function mapping everything to 5.)

2.4 Strings

Reproduced below are Exercises 6.1.6 and 6.1.7 of the Course Notes. Unlike the problems in previous sections above, I *will* test you directly on the techniques used to do these two exercises. You will be graded on the clarity, logical organization, and precision of your proofs.

13. Give inductive proofs of the following for all strings x, y, and z. You may assume without proof standard facts about natural numbers.

   (a) $|x| \geq 0$.

   (b) $|xy| = |x| + |y|$.

   (c) (Right Cancellation.) If $xz = yz$, then $x = y$.

   (d) (Left Cancellation.) If $xy = xz$, then $y = z$.

   Answer: We will assume all strings are over some arbitrary alphabet Σ.

   (a) *Proof.* By induction on x.

   Base Case: $x = \varepsilon$. Then $|x| = |\varepsilon| = 0$ by definition. Thus $|x| = 0 \geq 0$.

   Inductive Case: $x \neq \varepsilon$. Let $u \in \Sigma^*$ be the principal prefix of x and let $a \in \Sigma$ be the last symbol of x (so that $x = ua$). Assume (inductive hypothesis) that the statement holds for u, i.e., $|u| \geq 0$. Then by definition of length, $|x| = |u| + 1 \geq 0 + 1 = 1 \geq 0$. [The inductive hypothesis was used for the first $\geq$-relation.]

   Thus $|x| \geq 0$ for all strings x. □

   (b) *Proof.* By induction on y.

   Base Case: $y = \varepsilon$. Then $|y| = 0$ by definition. Then for any x, $|xy| = |x\varepsilon| = |x| = |x| + 0 = |x| + |y|$ (using the fact that $x\varepsilon = x$).

   Inductive Case: $y \neq \varepsilon$. Let $u \in \Sigma^*$ be the principal prefix of y and let $a \in \Sigma$ be the last symbol of y (so that $y = ua$). Assume (inductive hypothesis) that the statement holds for u, i.e., for any string x, $|xu| = |x| + |u|$. Then $|xy| = |xua| = |xu| + 1$ by definition of length. By the inductive hypothesis, $|xu| = |x| + |u|$, and so

$$|xy| = |xu| + 1 = |x| + |u| + 1 = |x| + |y| ,$$

   the last equality by the definition of length.

   Thus $|xy| = |x| + |y|$ for all strings x and y. □

   (c) *Proof.* By induction on z.

   Base Case: $z = \varepsilon$. Let x and y be arbitrary strings and suppose that $xz = yz$. Then

$$y = \varepsilon y = xy = yz = y\varepsilon = y ,$$

   so the statement holds when $z = \varepsilon$.

   Inductive Case: $z \neq \varepsilon$. Let $u \in \Sigma^*$ be the principal prefix of z and let $a \in \Sigma$ be the last symbol of z (so that $z = ua$). Assume (inductive hypothesis) that the statement holds for u, i.e., for any strings x and y, $xu = yu$ implies $x = y$. We prove the statement for z. For any given strings x and y, suppose that $xz = yz$. Then, since $z = ua$, we have $xua = yua$. By the uniqueness of the principal prefix, we must therefore have $xu = yu$. Then it follows by the inductive hypothesis that $x = y$. Thus the statement holds for z.

   Thus $xz = yz$ implies $x = y$ for all string x, y, and z. □

(d) This proof was harder and longer than I thought. I will not test you on a proof this involved.

Proof. This is by induction on y. We will use the general fact that, for all strings x, y, and z, if $xy = xz$ then $|y| = |z|$. This can be seen by using part (b) to get

$$|x| + |y| = |xy| = |xz| = |x| + |z|$$

then subtracting $|x|$ from both sides. We also use the fact that ε is the only string of length 0. To see this, we note that a nonempty string has length 1 greater than that of its principal prefix, which itself has length ≥ 0 by part (a).

Base Case: $y = \varepsilon$. Let x and z be arbitrary strings and suppose that $xy = xz$. Then $|y| = |z|$, and since $y = \varepsilon$ we have $|z| = |y| = 0$, which implies $z = \varepsilon$ as well, and thus $y = z$.

Inductive Case: $y \neq \varepsilon$. Let $u \in \Sigma^*$ be the principal prefix of y and let $a \in \Sigma$ be the last symbol of y (so that $y = ua$). Assume (inductive hypothesis) that the statement holds for u, i.e., for *any* strings x and v, $xu = xv$ implies $u = v$. We prove the statement for y. Suppose for any strings x and z that $xy = xz$. Then by the general fact above and the inductive definition of length (Definition 6.1.2 of the Course Notes), we have $|z| = |y| = |u| + 1 > 0$, whence $z \neq \varepsilon$. Now let w be the principal prefix of z and let b be the last symbol of z, so that $z = wb$. The equation $xy = xy$ becomes $xua = xvb$. The last symbol of xua is a, and the last symbol of xvb is b. Since these two strings are equal, their last symbols are equal (by the uniqueness of the last symbol), i.e., $a = b$. Similarly, the principal prefix of xua is xu and the principal prefix of xvb is xv. Since these two strings are equal, their principal prefixes are equal (by the uniqueness of the principal prefix), i.e., $xu = xv$. By the inductive hypothesis, $xu = xv$ implies $u = v$. We now have

$$y = ua = vb = z$$

as required.

Thus $xy = xz$ implies $y = z$ for all string x, y, and z. □

14. The *reversal* of a string x (denoted x^R) is the string formed by putting the symbols of x in reverse order. (For example, $(abcb)^R = bcba$.)

(a) Give a precise, inductive definition of the reversal x^R of a string x.

(b) Using your definition, give proofs by induction that $|x^R| = |x|$ and that $(x^R)^R = x$ for any string x.

Answer:

(a)

Definition 1. *For any string x, we define the string x^R inductively as follows:*

Base Case: $x = \varepsilon$. *Then define $x^R := \varepsilon$ (that is, $\varepsilon^R := \varepsilon$).*

Inductive Case: $x \neq \varepsilon$. *Then let y be the principal prefix of x and a the last symbol of x. Then define $x^R := a(y^R)$ (or dropping parentheses, $x^R := ay^R$, where the R applies only to the y).*

(b) We prove each statement separately in turn.

i. *Proof.* We prove that $|x^R| = |x|$ by induction on x.

Base Case: $x = \varepsilon$. Then $|x^R| = |\varepsilon^R| = |\varepsilon| = |x|$. [Both are 0 in this case, but that is not needed for the proof.]

Inductive Case: $x \neq \varepsilon$. Let $u \in \Sigma^*$ be the principal prefix of x and let $a \in \Sigma$ be the last symbol of x (so that $x = ua$). Assume (inductive hypothesis) that the statement holds for u, i.e., $|u^R| = |u|$. We prove the statement for x. By what we proved earlier (and the definition of string length given by Definition 6.1.2 of the Course Notes), we have

$$|x^R| = |(ua)^R| = |au^R| = |a| + |u^R| = |\varepsilon a| + |u^R| = |\varepsilon| + 1 + |u^R| = 1 + |u^R| \ .$$

Applying the inductive hypothesis, we have $|u^R| = |u|$, and so

$$|x^R| = 1 + |u| = |x| \ ,$$

the second equality following from the definition of length.

Thus $|x^R| = |x|$ for all strings x. □

ii. This proof was also harder than anticipated.

Proof. Before we prove that $(x^R)^R = x$ for any string x, we first prove for any string z and symbol a that

$$(az)^R = z^R a \tag{1}$$

by induction on z. For the base case $z = \varepsilon$, we have

$$\begin{aligned}
(az)^R = (a\varepsilon)^R = a^R = (\varepsilon a)^R & \\
= a\varepsilon^R & \quad \text{(by the inductive case of the def. of string reversal)} \\
= a\varepsilon & \quad \text{(by the base case of the def. of string reversal)} \\
= a = \varepsilon a & \\
= \varepsilon^R a & \quad \text{(by the base case of the def. of string reversal)} \\
= z^R a \ . &
\end{aligned}$$

For the inductive case $z \neq \varepsilon$, let $z = yb$, where b is the last symbol of z and y is the principal prefix of z. Assume (inductive hypothesis) that $(ay)^R = y^R a$. Then

$$\begin{aligned}
(az)^R &= (ayb)^R \\
&= b(ay)^R && \text{(by the inductive case of the def. of string reversal)} \\
&= by^R a && \text{(by the inductive hypothesis)} \\
&= (yb)^R a && \text{(by the inductive case of the def. of string reversal)} \\
&= z^R a \ .
\end{aligned}$$

This proves Eq. (**??**).
We now prove that $(x^R)^R = x$ by induction on x.

Base Case: $x = \varepsilon$. Then by the base case of the definition of string reversal above, we have $(x^R)^R = (\varepsilon^R)^R = \varepsilon^R = \varepsilon = x$.

Inductive Case: $x \neq \varepsilon$. Let $u \in \Sigma^*$ be the principal prefix of x and let $a \in \Sigma$ be the last symbol of x (so that $x = ua$). Assume (inductive hypothesis) that the statement holds for u, i.e., $(u^R)^R = u$. We prove the statement for x. Applying the inductive case of the definition of string reversal above twice, we have $(x^R)^R = ((ua)^R)^R = (au^R)^R$. Now we apply Eq. (**??**) above to get $(au^R)^R = (u^R)^R a$. Then by the inductive hypothesis, $(u^R)^R a = ua = x$.

Thus $(x^R)^R = x$ for all strings x. $\square$