The proofs of (2) and (3) are now trivial. The proof of (4) is also trivial.

\[ S = r^n + \cdots + r^1 + r^0 + 1 = \frac{r^n - 1}{r - 1} \]

Solve for \( S \).

\[
\frac{1 + r^n + \cdots + r^1 + r^0}{1 + r - 1} = S
\]

\[
\frac{p - 1}{1 + r^n - 1} = S
\]

**Proof.** We have

\[
\lim_{n \to \infty} |S|^{r^n} = r \quad \text{if} \quad 1 < |r|< \infty
\]

\[
\lim_{n \to \infty} |S|^{r^n} = c > 1 \quad \text{if} \quad |r| > 1
\]

\[
\lim_{n \to \infty} |S|^{r^n} = 0 \quad \text{if} \quad |r| < 1
\]

\[
\frac{p - 1}{1 + r^n - 1} = S
\]

**Theorem 3.1.** Let \( S \) be a geometric series.

\[ S = \frac{r^0 + \cdots + r^n}{1 - r} \]
are scanned and the number of active items is diminished by 1. Then the algorithm will take at least

Assume an algorithm starts with $n$ active items. With each iteration of an outer loop all the active items

$O(\mu^2)$ time in the worst case.
reason the time depends on \( n \), then the total time will be even worse than this. In the case of the bubble sort, each item is scanned in constant time and per item per iteration 12

\[
\frac{\text{per step}}{(u)^f} \cdot \frac{\text{per iteration}}{i - u} = \frac{u}{i^2}
\]

Note that the time could actually be greater than \( \Omega(u^2) \). The actual running time is

```c
/* end for */ {
  /* end for */ {
    {
      temp = [f]n;
      [f]n = [i]n;
      [i]n = temp;
    }
    ([f]n < [i]n) if (i)
      for (f) u; ++f; i = i + 1;
    for (i) u-1; ++i; t = u;
    steps;
  }
  /* end for */ {
    /* end for */ {
    }
  }
}
```

one. Thus in iterations (one for each of the items to be sorted), and the \( i \)-th iteration takes (worst case) \( u - i \) steps. One loop one item is put in its proper place and the number of elements remaining to be sorted drops by

The canonical example of such an algorithm is a bubble sort or insertion sort. With each iteration of the
doesn't at least look at the data in each iteration, and that leads to this recurrence.

Bottom line: Anything incremental almost has to be because you almost can't do an algorithm that

\[(\varepsilon u)O = \frac{z}{(u)(1-u)} = \sum_{i=0}^{\infty} (1-i)\]

have to the night. In this case, we have

We always carry the recursion until we get a set a least term, and then we look at what we

\[(I - u) + (z - u) + (\varepsilon - u) + \cdots + (z) + (1) + (1) L = \]

\[= \]

\[(I - u) + (z - u) + (\varepsilon - u) + (z - u) L = \]

\[(I - u) + (z - u) + (z - u) L = \]

\[(I - u) + (I - u) L = (u)L \]

Formally, we have
\[ \left( \frac{d^2 y}{u} \right) \int_y^y \sum_{I=1}^{I=I} + (I) L \cdot u = \]

\[ \left( \frac{d^2 y}{u} \right) \int_y^y \sum_{I=1}^{I=I} + (I) L \cdot u = \]

\[ \cdots = \]

\[ \left( \frac{d^2 y}{u} \right) \int_y^y + \frac{d}{u} \int y + (u) f + \left( \frac{d^2 y}{u} \right) L \cdot \psi \]

\[ (u) f + \left( \frac{d}{u} \int y + \left( \frac{d^2 y}{u} \right) L \cdot \psi \right) \cdot \psi = \]

\[ (u) f + \left( \frac{d^2 y}{u} \right) L \cdot \psi = (u) \]

Algorithm looks like this. recursively cuts the active list into pieces each of size \( n \). Then the running time of the overall Assume an algorithm starts with \( n \) active items. Each iteration of the outer loop takes time and
to produce any element in a merged list (except for the last one, which is free). The recurrence is thus

\[ 1 - u + \left( \frac{z}{u} \right) L \cdot z = (u)L \]

Merge sort has a recurrence that is
Where we have defined $\gamma \zeta = u$ by

$$(u \xi \iota u) O =$$

$I + u - u \xi u + (I) L \cdot u =$

$I + \gamma \zeta - u \cdot \gamma + (I) L \cdot u =$

$$\gamma \zeta \bigotimes_{0 \neq i} - u \bigotimes_{0 \neq i} + (I) L \gamma \zeta =$$

$$\cdots =$$

$I - u + \zeta - u + \varphi - u + \left(\frac{\gamma \zeta}{u}\right) L \cdot \varepsilon \zeta =$

$I - u + \zeta - u + \left(I - \frac{\varphi}{u} + \left(\frac{8}{u}\right) L \cdot \zeta\right) \cdot \varphi =$

$I - u + \zeta - u + \left(\frac{\varphi}{u}\right) L \cdot \varphi =$

$I - u + \left(I - \frac{\zeta}{u} + \left(\frac{\varphi}{u}\right) L \cdot \zeta\right) \cdot \zeta =$

$I - u + \left(\frac{\zeta}{u}\right) L \cdot \zeta = (u) L$
\((\varepsilon u)O = (I - u\tau) \cdot u\tau + (I)Lu = \)

\[
\left( \frac{I - \tau}{I - 1 + \gamma\tau} \right) \cdot 1 + \gamma\tau + (I)Lu = \]

\[
(1\tau + \cdots + 1 - \gamma\tau + \gamma\tau) \cdot 1 + \gamma\tau + (I)Lu = \]

\[
(1 - \gamma) - \gamma\tau + \cdots + \gamma - \gamma\tau + 1 - \gamma\tau + \gamma\tau + (I)Lu = \]

\[
\varepsilon(1\tau + \cdots + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon(1 - \gamma)\tau + \varepsilon(\gamma) + (I)Lu = \]

\[
\varepsilon(\varepsilon - 1\tau + \cdots + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon\varepsilon + (I)Lu = (u)L \]

If we let \(\gamma\tau = u\), then this eventually becomes:

\[
\varepsilon u + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon\varepsilon + (I)Lu = \]

\[
\varepsilon u + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon\varepsilon + (I)Lu + (\varepsilon u)\varepsilon = \]

\[
\varepsilon u + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon(\varepsilon - 1\tau)\tau + \varepsilon\varepsilon + (I)Lu + (\varepsilon u)\varepsilon = \]

\[
\varepsilon u + \left( \varepsilon \left( \frac{\varepsilon}{u} \right) + \left( \frac{\varepsilon}{u} \right) L \cdot \varepsilon \right) \cdot \varepsilon = \]

\[
\varepsilon u + \left( \varepsilon \left( \frac{\varepsilon}{u} \right) + \left( \frac{\varepsilon}{u} \right) L \cdot \varepsilon \right) \cdot \varepsilon = \]

\[
\varepsilon u + \left( \frac{\varepsilon}{u} \right) L \cdot \varepsilon = (u)L \]

Recursion that divides the data evenly into two halves?

We can't deal with data items in less than \(n\) steps. But what if the cost is much higher, with the same
\[(u \circ \mathcal{I} u)O = \]
\[
\left( \frac{z}{x+y}\right) u + (I)L u = \]
\[
(0 + (I) + \cdots + (I-y) + y) u + (I)L u = \]
\[
(z \circ \mathcal{I} + \cdots + (z/u) \circ \mathcal{I} + u \circ \mathcal{I}) u + (I)L u = (u)L \]

If we let \( z = u \), then this eventually becomes:
\[
\frac{\partial}{\partial x} u + (z/u) \frac{\partial}{\partial z} u + \left( \frac{\frac{\partial z}{\partial u}}{u} \right) L \cdot \zeta = \]
\[
\frac{\partial}{\partial x} u + (z/u) \frac{\partial}{\partial z} u + \left( \frac{\partial z}{\partial u} \right) \frac{\partial}{\partial u} \cdot \zeta = \]
\[
\frac{\partial}{\partial x} u + (z/u) \frac{\partial}{\partial z} u + \left( \frac{\partial z}{\partial u} \right) L \cdot \zeta = \]
\[
\frac{\partial}{\partial x} u + \left( \frac{z}{u} \right) L \cdot \zeta = (u)L \]

What about something in between? We have 
\[
\zeta \circ \partial = (u)f \quad \text{and} \quad (u \circ \mathcal{I} u)O \quad \text{leading to} \quad u = (u)f \quad \text{and} \quad (u \circ \mathcal{I} u)O \quad \text{leading to} \quad u = (u)f \]
\[(z+1)O = \]
\[
\left( \frac{2-z - I}{\gamma (2-z) - I} \right)_{z+1} u + (I) Lu = \]
\[
(1 + \gamma (2-z) + \cdots + 1 - \gamma (2-z) )_{z+1} u + (I) Lu = \]
\[
(\gamma (2/I) + \gamma (\mathcal{Z}/I) + \cdots + \gamma (1-\gamma \mathcal{Z}/I) )_{z+1} u + (I) L\mathcal{Z} = (u)L \]

If we let \( u = \frac{\mathcal{Z}}{\mathcal{V}} \), then this eventually becomes
\[
z+1 u + \gamma (\mathcal{Z}/I)_{z+1} u + \gamma (\mathcal{V}/I)_{z+1} u + \gamma (8/I)_{z+1} u + \left( \frac{\mathcal{Z}}{u} \right) L \cdot \mathcal{Z} = \]
\[
z+1 u + \gamma (\mathcal{Z}/u)\mathcal{Z} + \gamma (\mathcal{V}/u)\mathcal{V} + \gamma (8/u)\mathcal{8} + \left( \frac{\mathcal{Z}}{u} \right) L \cdot \mathcal{Z} = \]
\[
z+1 u + \gamma (\mathcal{Z}/u)\mathcal{Z} + \gamma (\mathcal{V}/u)\mathcal{V} + \left( \frac{\mathcal{V}}{u} \right) L \cdot \mathcal{Z} = \]
\[
z+1 u + \gamma (\mathcal{Z}/u)\mathcal{Z} + \left( \frac{\mathcal{V}}{u} \right) L \cdot \mathcal{V} = \]
\[
z+1 u + \left( \gamma (\mathcal{Z}/u) + \left( \frac{\mathcal{V}}{u} \right) L \cdot \mathcal{Z} \right) \cdot \mathcal{Z} = \]
\[
z+1 u + \left( \frac{\mathcal{Z}}{u} \right) L \cdot \mathcal{Z} = (u) L \]
\[(b^u)O = (u)_L < b \int \exists \cdot \]

\[(y^u)O = (u)_L \exists > b \int \exists \cdot \]

\[(u \log b^u)O = (u)_L \exists = b \int \exists \cdot \]

\[\text{Theorem } 3.2 \text{ Let } \exists \]

\[\text{for constants } A, B, C, \text{ and } K.\]
\[(u \otimes_g u) O = u \otimes_g u \mathcal{C} + (1) \mathcal{L}_g u = u \otimes_g u \mathcal{C} + (1) \mathcal{L}_g V = (u) \mathcal{L}\]

If we then have \( \gamma = \mathcal{B} \), then we have \( \mathcal{B} \), so we have

\[
\left( I + (\gamma V / g V) + \gamma (\gamma V / g V) + \cdots + \gamma (\gamma V / g V) + (I - \gamma (\gamma V / g V)) \right) \gamma u \mathcal{C} + (1) \mathcal{L}_g V = \\
\left( I + \gamma (V / I) g V + \gamma (\gamma V / I) g \gamma V + \cdots + \gamma (I - \gamma V / I) g (I - \gamma) V \right) \gamma u \mathcal{C} + (1) \mathcal{L}_g V = \\
\left( \gamma u + \gamma (V / u) g V + \gamma (\gamma V / u) g \gamma V + \cdots + \gamma (I - \gamma V / u) g (I - \gamma) V \right) \mathcal{C} + (1) \mathcal{L}_g V = \\
\cdots = \\
\left( \gamma u + \gamma (V / u) g V + \gamma (\gamma V / u) g \gamma V \right) \mathcal{C} + (\gamma V / u) \mathcal{L}_g V = \\
\left( \gamma u + \gamma (V / u) g V \right) \mathcal{C} + (\gamma V / u) \mathcal{L}_g V = \\
\gamma u \mathcal{C} + \gamma (V / u) \mathcal{C} + (\gamma V / u) \mathcal{L}_g V = \\
\gamma u \mathcal{C} + (\gamma (V / u) \mathcal{C} + (\gamma V / u) \mathcal{L}_g V) g V = \\
\gamma u \mathcal{C} + (V / u) \mathcal{L}_g V = (u) \mathcal{L}
\]

**Proof.** With the usual eventual definition of \( \gamma = u \otimes \mathcal{C} \), or \( \gamma V = u \otimes \mathcal{C} \), we have
and we're done.

\[(u - g u) \left( \frac{1 - y - g V}{C} \right) + (I)L g u = (u)L\]

If \( y < B \), then we have:

\[ (u - g u) \left( \frac{1 - y - g V}{C} \right) + (I)L g u = (u)L \]

If \( y > B \), then \( u \) is a constant, so we have:

\[ (I - y - g u) \left( \frac{1 - y - g V}{C} \right) + (I)L g u = \]

\[ \left( \frac{y - g V - 1}{y(y - g V) - 1} \right) u c + (I)L g u = \]

\[ (I + (y - g V) + (y - g V) + \cdots + (y - g V) + (y - g V)) u c + (I)L g u = \]

\[ (I + (y V/g V) + (y V/g V) + \cdots + (y V/g V) + (y V/g V)) u c + (I)L g V = (u)L \]
\[(u f') \Theta = (u) \mathcal{L}\]

for some constant \( c > 1 \) and all sufficiently large \( n \), then

\[(u) f' \Theta \leq (q/u) f' \nu\]

\[\forall \varepsilon > 0 \quad \exists \nu \text{ constant } \varepsilon\]

\[(u \tilde{f} (u)f') \Theta = (u \tilde{f} (u)\tilde{f}^{\nu} u) \Theta = (u) \mathcal{L}\]

then \((u \tilde{f}^{\nu} u) \Theta = (u) f \mathcal{L} \cdot \varepsilon\]

\[(u \tilde{f}^{\nu} u) \Theta = (u) \mathcal{L} \]

then, for some constant \( \varepsilon > 0 \),

\[\exists (\varepsilon - (\nu) \tilde{f}^{\nu} u) \mathcal{O} = (u) f \mathcal{L} \cdot I. \]

We assume that either \((q/u) \mathcal{L} = q / u \) or \([q/u] = q / u \) holds, then the following hold

\[\exists (u) f + (q/u) \mathcal{L} \nu = (u) \mathcal{L}\]

for \( q \in \mathbb{Z} \)

\[\text{Theorem 3.3. Let } a < q < 1 \text{ and } \exists \mathcal{L} \text{ be a constant, and let } f \text{ be a function, and let } \mathcal{L} \text{ be defined on}\]

\[\text{Theorem 3.6. The Master Theorem}\]
\[ u + u \varepsilon + u_6 + \cdots + u_{1-\varepsilon} + \left( \frac{\varepsilon}{u} \right) L\varepsilon 6 = \]
\[ \cdots = \]
\[ u + u \varepsilon + u_6 + \left( \frac{\varepsilon}{u} \right) L_6 \varepsilon L = \]
\[ u + u \varepsilon + \left( \frac{6}{u} + \left( \frac{\varepsilon}{u} \right) L_6 \right) 18 = \]
\[ u + u \varepsilon + \left( \frac{6}{u} \right) L18 = \]
\[ u + \left( \frac{\varepsilon}{u} + \left( \frac{6}{u} \right) L6 \right) 6 = \]
\[ u + \left( \frac{\varepsilon}{u} \right) L6 = (u)L \]

Now let's do this the direct way,

\[ (\varepsilon/u) L = (u)L \]

We can let \( \varepsilon = 1 \) in case 1 of the theorem, and get

\[ u + (\varepsilon/u) L6 = (u)L \]
\((u \log) \Theta = (u)_L\)

Case 2 applies, so

\[ I = 0 \, u = (1)_{\log} u = (u)_{\log} u \]

Using the master theorem, we have \( a = 1 \) and

\[ I + (\varepsilon/\log u)_L = (u)_L \]

**Example 2**

Since we multiply this term by \( u \), the entire recurrence is

\[ (u)_O \]

Now, since we do the recurrence until \( u \), this is

\[ \frac{\varepsilon}{1 - \varepsilon} \]

which equals (it's a geometric sum)

\[ 1 + \cdots + \varepsilon + 6 + \varepsilon + 1 \]

We do this until \( u = \varepsilon \), at which point the \( I \) term is \( u \). The rest
\[ u \otimes u + (\forall / u) \varepsilon = (u)_L \]

**Example 3**

and asymptotically all logs are the same.

\[ u \varepsilon / \varepsilon \log = \gamma \]

And if we do the recursion that far, then

\[ \gamma (\varepsilon / \varepsilon) = u \]

where we do the recursion until

\[ \gamma + (\gamma (\varepsilon / \varepsilon) \cdot u)_L = \]

\[ \ldots = \]

\[ 1 + 1 + I + (\varepsilon / u \varepsilon)_L = \]

\[ I + I + (\varepsilon / u \forall)_L = \]

\[ I + (\varepsilon / u \varepsilon)_L = (u)_L \]

And doing this the direct way...
So the lead term can be ignored because it will be dominated by the term at the right hand end.

We can thus until \( u \leq \xi \), at which point the term is a constant multiple of \( \xi \), that is less than \( u \).

\[
\begin{align*}
  u \triangleright u + (\psi/u) \triangleright (\psi/u^2) + \cdots + (1 - \psi \psi / u) \triangleright u_{1 - \psi} (\psi / \xi) + (\psi \psi / u) L \xi \xi &= \\
  \cdots &= \\
  u \triangleright u + (\psi/u) \triangleright (\psi/u^2) + (\psi \psi / u) L \xi &= \\
  u \triangleright u + (\psi/u, \xi) \geq (\psi/u) \triangleright (\psi/u) \xi = (q/u) f \psi
\end{align*}
\]

The direct way:

\[
(u \triangleright u) \theta = (u) L
\]

For \( c = \psi / u \), hence

\[
(u) f c = u \triangleright u (\psi / \xi) \geq (\psi/u) \triangleright (\psi/u) \xi = (q/u) f \psi
\]

We could choose \( z \approx \varepsilon \) and use 

\[
O \cdot e^{\lambda_{1} q} u = e^{\lambda_{1} q} u = e^{\lambda_{1} q} u
\]

And for \( n \) sufficiently large, we have \( \psi = q \), \( \xi = 0 \).
In order to apply the theorem we need that it be polynomially larger, which it isn't. We have \( u \not\leq u \) and \( u \not\leq u \) asymptotically larger than

\[
u \not\leq u + (\varepsilon/u) L \geq (u) L
\]

And now one where the method doesn't work.

**Example 4**

\[
u \not\leq u \cdot \text{constant} \subseteq
\]

\[
(\varepsilon (\varepsilon / \varepsilon) - 1) \cdot \varepsilon \cdot (u \not\leq u) =
\]

\[
\left(\frac{\varepsilon}{\varepsilon}\right) \sum_{1-\gamma} u \not\leq u \subseteq
\]

\[
(\left(\frac{\varepsilon}{\varepsilon}\right) \sum_{1-\gamma} u \not\leq u) \cdot \left(\frac{\varepsilon}{\varepsilon}\right) \sum_{1-\gamma} u =
\]

\[
(\sum_{1-\gamma} u) \cdot (\varepsilon / \varepsilon) \sum_{1-\gamma} u = (\sum_{1-\gamma} u) \cdot (\varepsilon / \varepsilon) \sum_{1-\gamma} u
\]

From above the sum clearly this is not \( u \not\leq u \) because the right hand term is and the rest are positive. So let's look at bounding...
Getting the same answer two different ways should make you feel a little more confident that have the right answer.

But it is also good to be able to check the results directly to make sure that you have applied the theorem correctly.

It is easier to do these if you can remember and properly apply the Master Theorem.