FINITE TIME CONTROL

HIGH-RATE CHALLENGE MONTHLY MEETING

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BACKGROUND

- Joined USC in Fall 2022
  - Department of Mechanical Engineering
    - Aerospace Engineering program

- Research interest
  - Finite time stability and control
  - Autonomous multi-agent system
  - Swarm intelligence optimization methods
  - Long-term autonomous mission using UAVs
  - Satellite constellation

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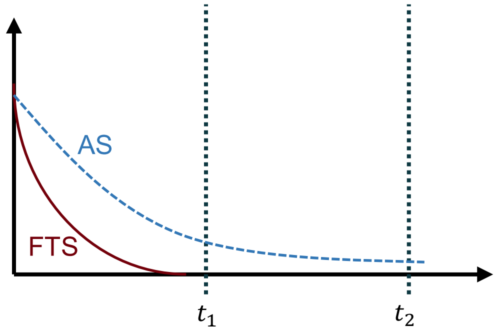
OUTLINE

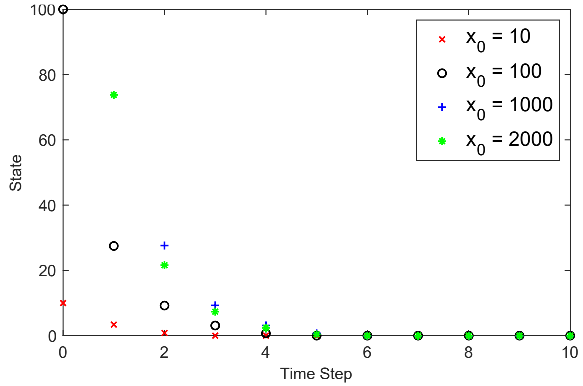
- Finite time stability: Deterministic system
  - Stability notions in finite time window
  - Gap between continuous-time and discrete-time dynamical systems
  - Digital implementation of finite time control
  - Extensions to nontangency analysis

- Stochastic finite time stability
  - Stochastic convergence
  - Error bound characterization

STABILITY NOTIONS IN FINITE TIME

- Finite time stability (FTS)
  - Lyapunov stability + Finite time convergence
  - Characterization of convergence time → upper bound
- Advantages of FTS
  - Robust against disturbances
  - Discontinuous system dynamics
  - Sequential mission
  - Faster convergence
- Fixed time stability (FxTS)
  - Finite time stable + Uniform boundedness of the convergence time





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GAP BETWEEN CT & DT

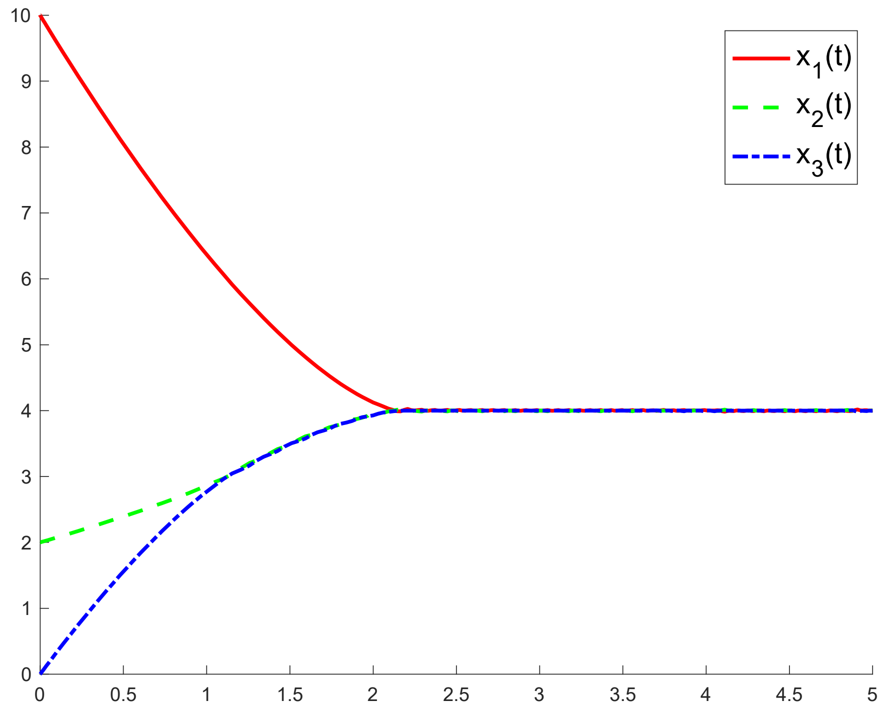
- Continuous-time (CT) system
  - Finite time convergence for higher dimension system
  - Characterization of convergence is more available
- Discrete-time (DT) system
  - Discrete-time theory requires vector-valued map
  - CT → DT requires 'smart' discretization to preserve finite time stability
    - Asymptotic stability
    - Finite time convergence
- Digital chattering

DIGITAL CHATTERING: EXAMPLE

- Three-dimensional CT system

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}\left(x_2(t)-x_1(t)\right)^{1/3}+\left(x_3(t)-x_1(t)\right)^{1/3}\\ \left(x_1(t)-x_2(t)\right)^{1/3}+\left(x_3(t)-x_2(t)\right)^{1/3}\\ \left(x_1(t)-x_3(t)\right)^{1/3}+\left(x_2(t)-x_3(t)\right)^{1/3}\end{bmatrix}$$

- Fully-connected three agents
  - Distributed architecture

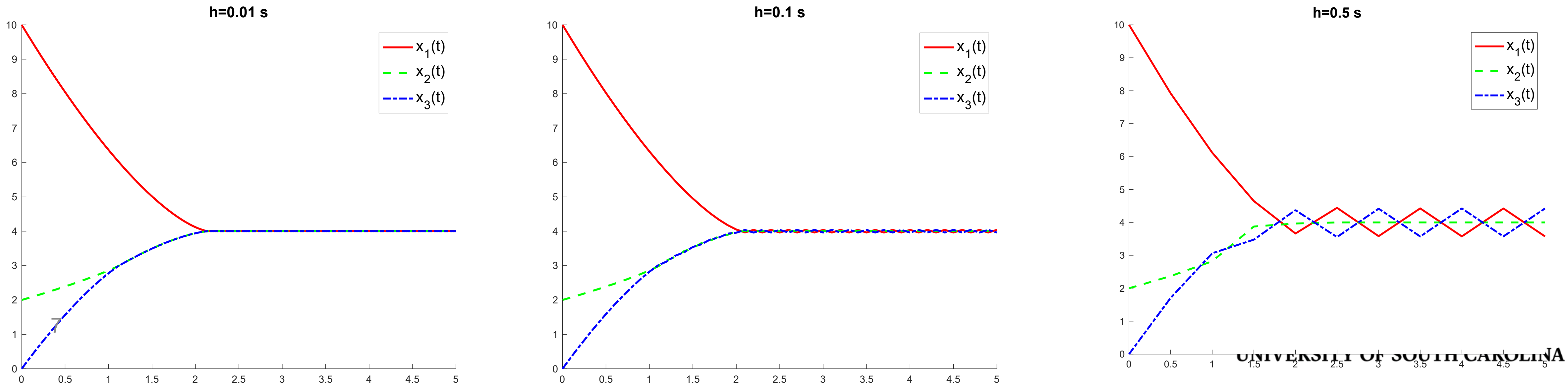


DIGITAL CHATTERING: EXAMPLE

- Explicit discretization

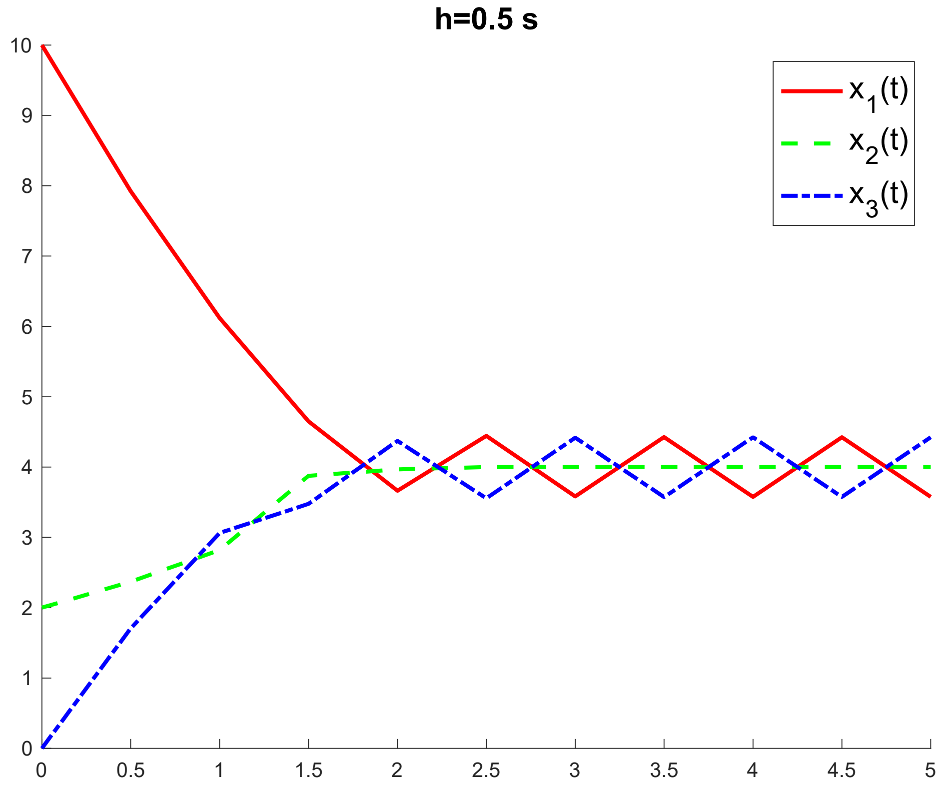
$$\Delta x(t) = \begin{bmatrix} \left(x_2(t) - x_1(t)\right)^{1/3} + \left(x_3(t) - x_1(t)\right)^{1/3} \\ \left(x_1(t) - x_2(t)\right)^{1/3} + \left(x_3(t) - x_2(t)\right)^{1/3} \\ \left(x_1(t) - x_3(t)\right)^{1/3} + \left(x_2(t) - x_3(t)\right)^{1/3} \end{bmatrix}$$

$$x(t+1) = x(t) + h\,\Delta x(t)$$



DIGITAL CHATTERING: EXAMPLE

- Lyapunov stability fails for smaller sampling rate



HIGH-RATE SAMPLING

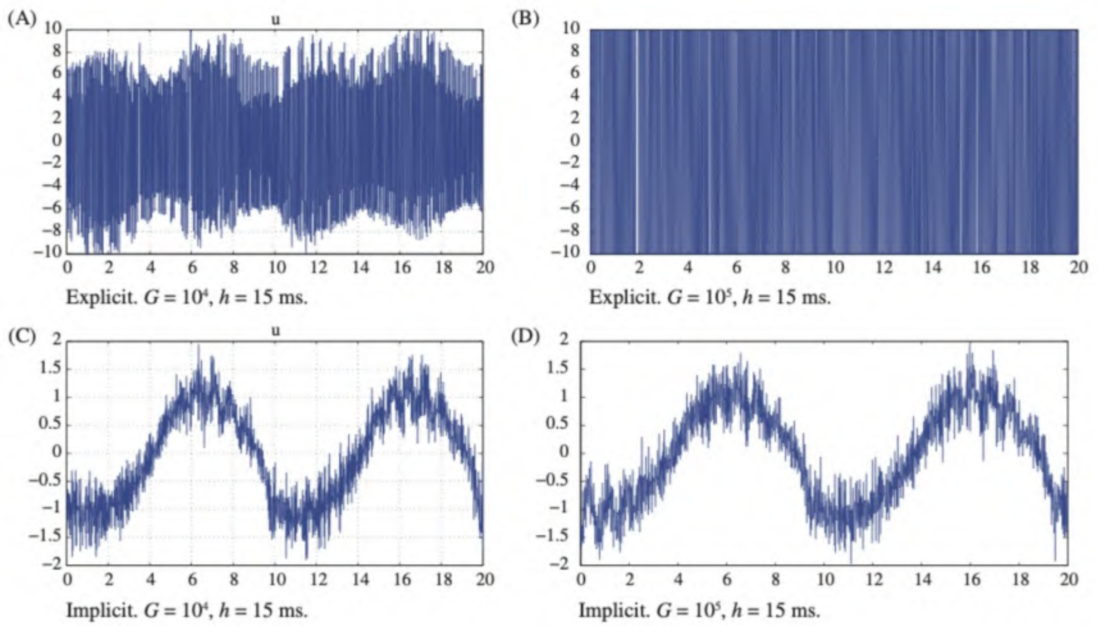
- High-rate sampling
  - Higher rate sampling time guarantees small error

- Challenges
  - The relationship between sampling rate and minimum error-bound
  - Characterization of convergence to achieve error-bound finite-time convergence
  - High-rate sampling might be expensive
  - High-dimensional system

IMPLICIT DISCRETIZATION

- Smart discretization method
  - Preserves stability and FT convergence

- Implicit method
  - What would be the hyper parameter?
  - How can we use system structure?
  - Distributed finite time control?
  - Connection to sliding mode controller



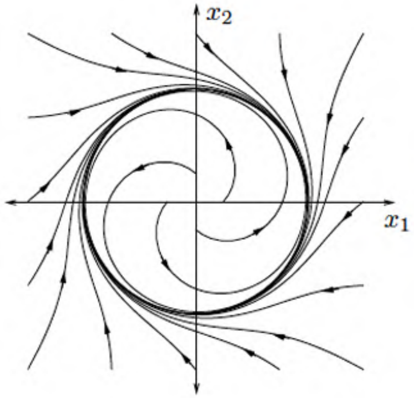
Credit: B. Brogliato, A. Polyakov, “Digital implementation of sliding-mode control via the implicit method: A tutorial”

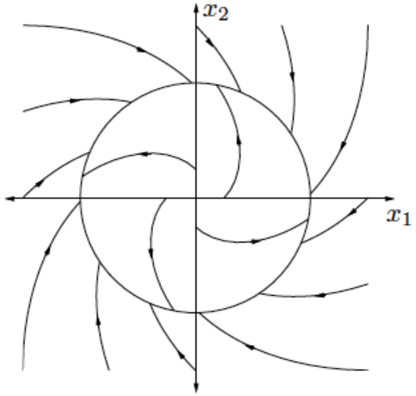
NONTANGENCY OF THE VECTOR FIELD

- Vector field nontangency analysis
  - Direction cone & tangent cone
  - Limiting direction of the vector field
  - Decomposable system

- Does not require the positive definite Lyapunov function
  - Less restrictive Lyapunov function

- Convergence test and semistability analysis
  - Guarantees stability and finite time convergence





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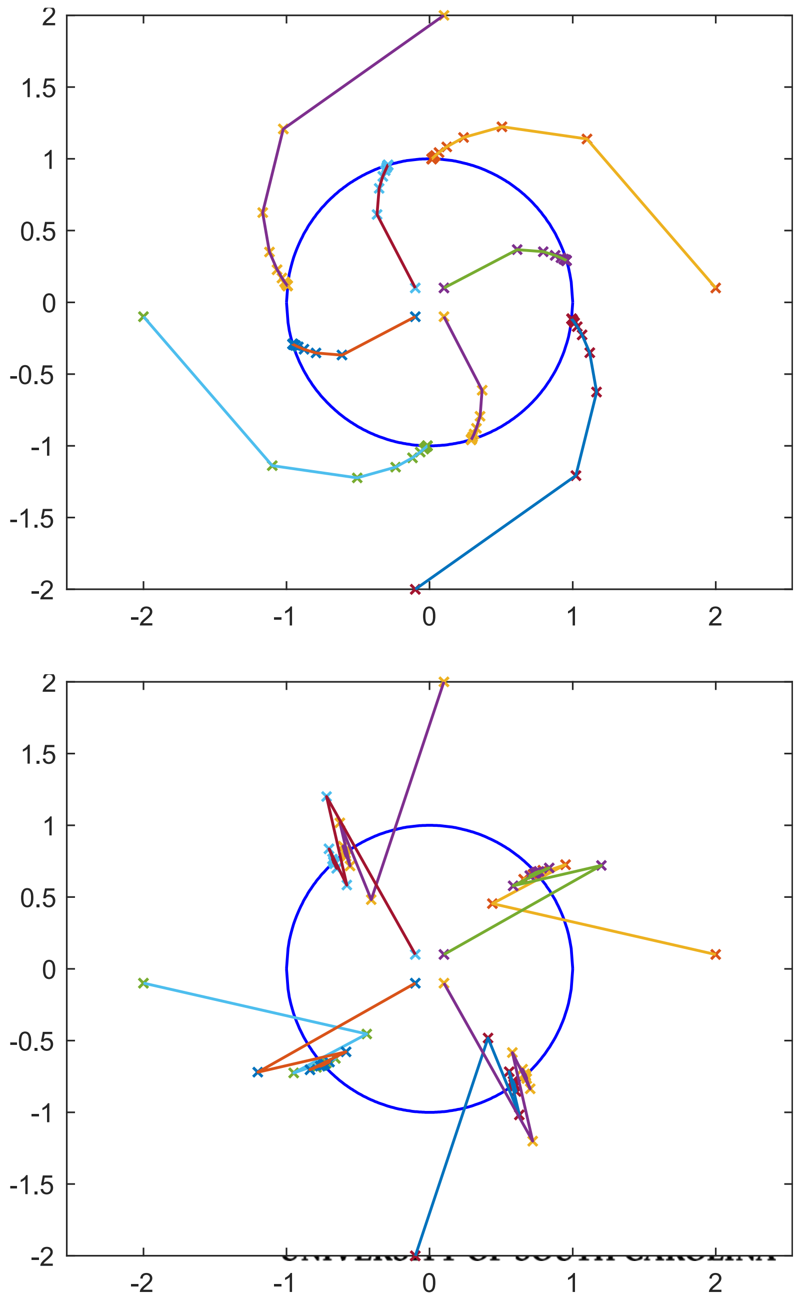
NONTANGENCY ANALYSIS: DECOMPOSABLE SYSTEM

- Decomposable system

$$x(k+1) = f\big(x(k)\big)$$

  - Given $k > 1$, $f^k\colon \mathcal{D} \to \mathcal{D}$ denotes the k-fold composition of f with itself while Δf^k denotes the map $x \mapsto f^k(x) - x$
  - Given $K \in \mathbb{Z}_+$, a K-fold decomposition for the system is a tuple $\mathfrak{D}$ consisting of K positive integers $k_1, \dots, k_K$ and K disjoint sets $\mathcal{D}_1, \dots, \mathcal{D}_K \subseteq \mathcal{D} \setminus \Delta \mathrm{f}^{-1}(0)$ such that $\cup_{i=1}^{K} \mathcal{D}_i = \mathcal{D} \setminus \Delta f^{-1}(0)$ and $f^{k_i}(\mathcal{D}_i) \subseteq \mathcal{D}_i$ for each $i = 1, \dots, K$.



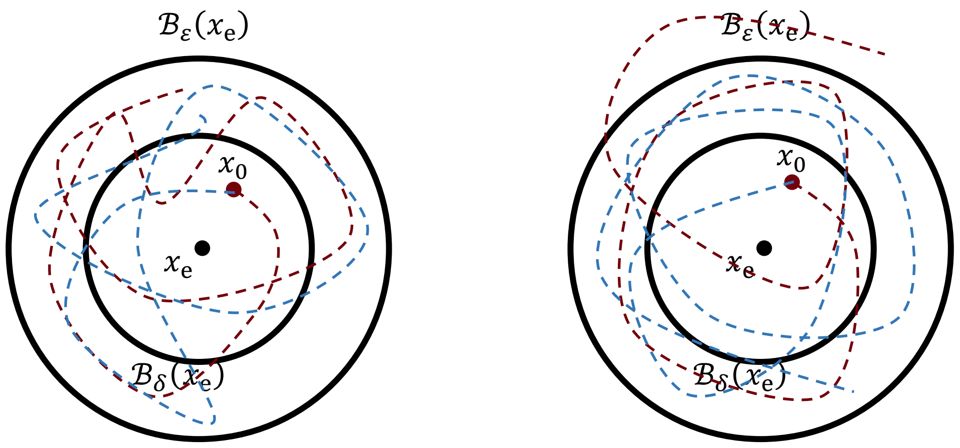
STOCHASTIC STABILITY: DISCRETE-TIME

- DT stochastic dynamical systems $\mathcal{G}$

$$x(k+1) = f\big(x(k)\big) + D\big(x(k)\big)w(k) \overset{\text{def}}{=} F\big(x(k), w(k)\big), \qquad x(0) = x_0, \qquad k \in \overline{\mathbb{Z}}_+$$

  - $x(k) \in \mathcal{D} \subseteq \mathbb{R}^n$: $\mathcal{D}$-valued stochastic process
  - $w(k)$: d-dimensional i.i.d. stochastic process
  - $f : \mathcal{D} \to \mathcal{D}$ & $D : \mathcal{D} \to \mathbb{R}^{n\times d}$
  - Equilibrium point x_e : $f(x_\mathrm{e}) = x_\mathrm{e}$ & $D(x_\mathrm{e}) = 0$

- Stability in probability: probability of escape
  - Lyapunov stability in probability
    - $\lim_{x_0 \to x_e} \mathbb{P}\left(\sup_{k\in\overline{\mathbb{Z}}_+} \|x(k) - x_e\| > \varepsilon\right) = 0$
  - Asymptotic stability in probability
    - $\lim_{x_0 \to x_e} \mathbb{P}\left(\lim_{k\to\infty} \|x(k) - x_e\| = 0\right) = 1$



FINITE TIME STABILITY IN PROBABILITY

- Stochastic settling time $K(x, \omega)$
  - State indexed stochastic process $K\colon \mathcal{D} \times \Omega \to \overline{\mathbb{Z}}_+$

- Finite time stability in probability
  - Lyapunov stability in probability
  - Finiteness of the stochastic settling-time
    - The stochastic settling-time $K(x, \cdot)$ is finite almost surely
  - Finite-time convergence in probability
    - $\mathbb{P}\left(\left\|s^{x_0}\left(K(x_0, \omega)\right) - x_{\mathrm{e}}\right\| = 0\right) = 1$

Probability of escape

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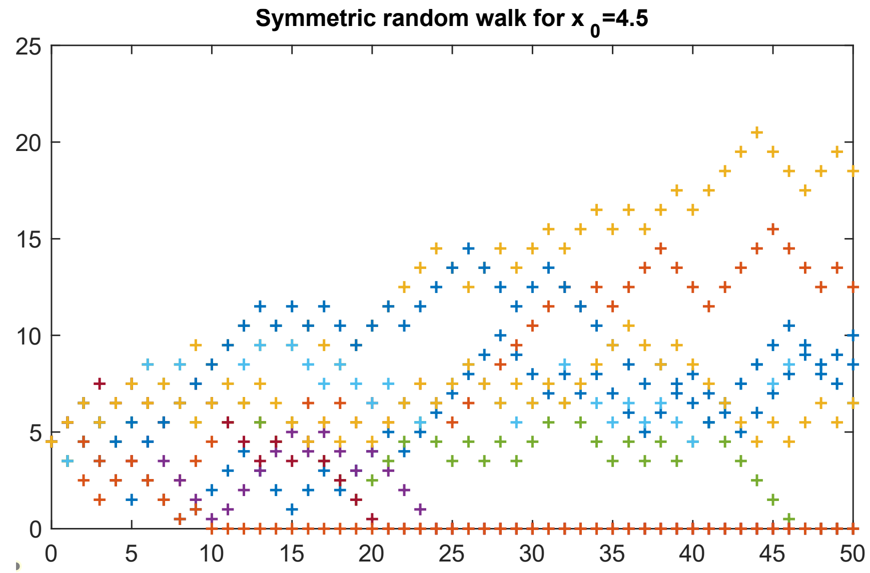
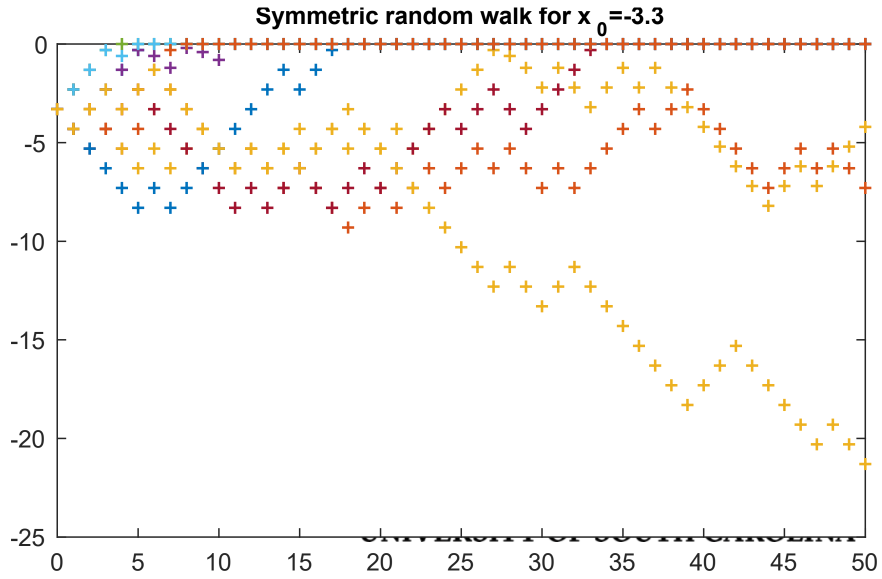
FINITENESS OF SETTLING TIME

- How do we define stochastic FT stability notion?
  - Finiteness of stochastic settling time
    - $K(x,\omega) < \infty$ almost surely
    - More general definition (E.g. symmetric random walk)
    - Hard to check
  - Finiteness of the expectation of stochastic settling time
    - $\mathbb{E}[K(x,\omega)] < \infty$
    - Narrow definition (E.g. symmetric random walk)
    - Easy to check





ERROR-BOUND FINITE TIME SEMISTABILITY

- Characterization of convergence rate
  - Upper bound for the convergence time to the given error-bound
  - Exponential (Geometric) semistability
- Complex dynamic behavior at the exact convergence
  - CT : Non-Lipschitzian dynamics
  - DT: Discontinuous state jump
- Distributed control architecture for multi-agent network system
  - Thermodynamic based consensus protocol
- Upcoming plan
  - Currently working on NSF proposal: Planned to be submitted in Spring 2023
  - Under review for IEEE L-CSS and CDC

DIGITAL IMPLEMENTATION OF FT CONTROLLER

- Bridge the gap between CT and DT FT controller design
  - CT finite time control
  - Insights from discrete time system on difference inclusion
  - Implicit discretization method
  - Selection of discretization parameters

- Application
  - High-rate sampling: impulsive dynamics, structural control
  - Low-rate sampling: spacecraft

SUMMARY

- Concept of FT stability and control
- Gap between CT and DT dynamical systems
- Digital implementation of FT controller
- Stochastic framework for FT controller
- Future application of FT controller